EXERCISE 6-1 Prove that the quotient and remainder ($q(x)$ and $r(x)$) are unique for each pair $(f(x), g(x))$.

There is a special shorthand method called **synthetic division** for dividing polynomials by expressions of the form $(x - a)$. To introduce synthetic division, we'll take you step by step through a problem which will be solved with both long division and synthetic division. Pay close attention not only to how to perform synthetic division, but also why it works.

$$
\begin{array}{r|rrrrrrr l}
 & x^2 & + & 3x & + & 2 & & & (1) \\
x-1 & x^3 & + & 2x^2 & - & x & + & 3 & (2) \\
 & -x^3 & + & \mathbf{1}x^2 & & & & & (3) \\
 & & & 3x^2 & - & x & + & 3 & (4) \\
 & & & -3x^2 & + & \mathbf{3}x & & & (5) \\
 & & & & & 2x & + & 3 & (6) \\
 & & & & & -2x & + & \mathbf{2} & (7) \\
 & & & & & & & 5 & (8)
\end{array}
$$

Above we did the long division of $x - 1$ into $f(x) = x^3 + 2x^2 - x + 3$. For synthetic division (shown below), we don't write any x's. The 1 from the constant term of $x - 1$ goes to the left of the vertical line on line (9). The coefficients of $f(x)$ are then copied into the remainder of that line. Line (11) represents the coefficients of the quotient. Clearly the first such coefficient is the first coefficient of $f(x)$ (since the leading coefficient in $(x - 1)$ is 1). Hence, we copy the first coefficient of $f(x)$ in line (9) into line (11). Now we have to figure out how to get the rest of line (11).

$$
\begin{array}{r|rrrr l}
1 & 1 & 2 & -1 & 3 & (9) \\
 & & \mathbf{1} & \mathbf{3} & \mathbf{2} & (10) \\
\hline
 & 1 & 3 & 2 & 5 & (11)
\end{array}
$$

Line (10) represents the subtractions at lines (3), (5), and (7) in the long division. In the long division, we get these by subtracting the product of the quotient and $x - 1$. Since the first term in the long divisions on these three lines always cancel, we are only interested in the second terms (the boldface coefficients). These results are from multiplying $-(-1)$ by the quotient (line (1)). (The first negative comes from the fact that we are *subtracting* the products of the quotient and $x - 1$ on lines (3), (5), and (7).) The coefficients of the quotient are on line (11), so we get line (10) from multiplying line (11) and the 1 at the left of our vertical line.

Finally, how do we determine line (11), the quotient? Since the leading coefficient of $(x - 1)$ is one, the coefficients of the quotient are the coefficients of the leading terms resulting from the combinations of lines (2) and (3), lines (4) and (5), and lines (6) and (7). Note that these are just the sums of the boldface numbers and the coefficients of the original $f(x)$! Hence, we get line (11) from just adding lines (9) and (10).

Here's how synthetic division works in action. We'll divide $x - 2$ into $x^3 - 3x^2 + 7x + 4$. First we copy the 2 from the constant term of $x - 2$ and the coefficients of $x^3 - 3x^2 + 7x + 4$ into our table. Then, we copy the first coefficient into line (3):

$$
\begin{array}{r|rrrr l}
2 & 1 & -3 & 7 & 4 & (1) \\
 & & & & & (2) \\
\hline
 & 1 & & & & (3)
\end{array}
$$

We now get the first number in line (2) by multiplying the 2 to the left of the vertical line and the 1 in line (3). After this, we add the number in line (2) to the number above it to get the next coefficient of the quotient in line (3):

$$\begin{array}{r|rrrrl} 2 & 1 & -3 & 7 & 4 & (1) \\ & & 2 & & & (2) \\ \hline & 1 & -1 & & & (3) \end{array}$$

We continue by multiplying our 2 and the next term in line (3), −1, to get the next term in line (2):

$$\begin{array}{r|rrrrl} 2 & 1 & -3 & 7 & 4 & (1) \\ & & 2 & -2 & & (2) \\ \hline & 1 & -1 & 5 & & (3) \end{array}$$

Now we can finish off the problem by getting our last terms in lines (2) and (3):

$$\begin{array}{r|rrrrl} 2 & 1 & -3 & 7 & 4 & (1) \\ & & 2 & -2 & 10 & (2) \\ \hline & 1 & -1 & 5 & 14 & (3) \end{array}$$

So what's the answer? The last line gives us the coefficients of $x^2 - x + 5$, but what's the 14 for? Compare synthetic division to long division and you'll find that 14 is the remainder, so the above synthetic division tells us that

$$\frac{x^3 - 3x^2 + 7x + 4}{x - 2} = x^2 - x + 5 + \frac{14}{x-2}.$$

There are a couple of important points to remember when doing synthetic division. First, it only works when we are dividing by a linear polynomial $(x - a)$. Second, the leading coefficient of this linear term must be 1. (Look at our development of synthetic division to see why we can't use synthetic division with a linear coefficient other than 1.) Finally, in synthetic division the term to the left of the vertical line is the negative of the constant term of the linear divisor. For example, in the above problem where we divided $x - 2$ into $x^3 - 3x^2 + 7x + 4$, we put 2, not −2, at the left of the vertical line.

EXAMPLE 6-1 Use synthetic division to determine $(8x^4 - 12x^3 + 2x + 1)/(2x + 1)$.

Solution: First, we must make the coefficient of x in the divisor 1. Hence, we divide the numerator and denominator by 2 to get

$$\frac{4x^4 - 6x^3 + x + 1/2}{x + 1/2}.$$

Now we do our synthetic division:

$$\begin{array}{r|rrrrr} -1/2 & 4 & -6 & 0 & 1 & 1/2 \\ & & -2 & 4 & -2 & 1/2 \\ \hline & 4 & -8 & 4 & -1 & 1 \end{array}$$

(Why is there a 0 in the first line above?) Thus, we find

$$\frac{4x^4 - 6x^3 + x + 1/2}{x + 1/2} = 4x^3 - 8x^2 + 4x - 1 + \frac{1}{x + 1/2},$$

so

$$\frac{8x^4 - 12x^3 + 2x + 1}{2x + 1} = 4x^3 - 8x^2 + 4x - 1 + \frac{2}{2x + 1}.$$

EXERCISE 6-2 Use synthetic division to divide $x + 3$ into $x^5 + 3x^4 + 2x^3 - x^2 + x - 7$.

6.3 Finding Roots of Polynomials

Suppose we are given the polynomial $f(x)$ and asked to find the solutions to $f(x) = 0$. We call these solutions **roots** of the polynomial. Unfortunately, no quick and easy method like the quadratic formula exists to solve general polynomials. Instead we must go searching for the roots. Does this mean that we just have to keep guessing values for x until we find one for which $f(x) = 0$? And how will we know if we've found all such x? Fortunately, we are not completely consigned to guessing. We do have some helpful hints to guide our way.

First, if a is a root, then $(x - a)$ divides $f(x)$ evenly; that is, there is no remainder when we perform the division. To see this we write

$$f(x) = (x - a)q(x) + r(x).$$

Since $\deg r(x) < \deg(x - a) = 1$, $\deg r(x) = 0$, and $r(x)$ is some constant c. Letting $x = a$ gives

$$f(a) = (a - a)h(a) + c = c.$$

If $f(a) = 0$, we have $c = 0$, and thus there is no remainder when we divide $f(x)$ by $(x - a)$.

EXAMPLE 6-2 Prove that the remainder upon dividing $f(x)$ by $x - a$ is $f(a)$.

Solution: As above, we write

$$f(x) = (x - a)h(x) + r(x).$$

Since $\deg r(x) < \deg(x - a)$, $r(x)$ is a constant r. Letting $x = a$ gives $f(a) = r$, so the remainder upon dividing $f(x)$ by $x - a$ is $f(a)$. Therefore, we can use synthetic division to determine $f(a)$ by finding the remainder when $f(x)$ is divided by $(x - a)$.

EXAMPLE 6-3 $P(x)$ is a polynomial with real coefficients. When $P(x)$ is divided by $x - 1$, the remainder is 3. When $P(x)$ is divided by $x - 2$, the remainder is 5. Find the remainder when $P(x)$ is divided by $x^2 - 3x + 2$. (MAΘ 1990)

Solution: We write

$$P(x) = (x^2 - 3x + 2)q(x) + r(x),$$

where $r(x)$ is the desired remainder. Since $\deg r(x) < \deg(x^2 - 3x + 2)$, we can write $r(x) = ax + b$ for some constants a and b. From the given information, we know $P(1) = 3$ and $P(2) = 5$. Since $x^2 - 3x + 2 = 0$ for $x = 2$ and $x = 1$, we put these values in our equation for $P(x)$, yielding

$$(0)q(1) + r(1) = a + b = P(1) = 3$$

$$(0)q(2) + r(2) = 2a + b = P(2) = 5.$$

Solving this system, we find $(a, b) = (2, 1)$, so the remainder is $\mathbf{2x+1}$. Remember this method of cleverly choosing values for x in polynomial equations; it can be very useful!

The **Fundamental Theorem of Algebra** states that every polynomial has at least one root. Thus, there is at least one value a such that $f(a) = 0$. This a may be real, imaginary, rational, or irrational, but the Fundamental Theorem of Algebra assures us that at least one such root exists. Unfortunately the proof is a bit too complex for this text, but we shall put the theorem to good use by showing that any degree n polynomial has exactly n roots. This means we can write any polynomial $f(x)$ as

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = a_n(x - r_1)(x - r_2)\cdots(x - r_n).$$

The r_i are the roots of the polynomial and they are not necessarily real or rational. It should be clear why $f(r_i) = 0$.

To show that all polynomials can be written in such a fashion we invoke the Fundamental Theorem of Algebra. By this theorem, we know that for some number r_1 we can write

$$f(x) = (x - r_1)q_1(x).$$

Since $\deg f = n = \deg[(x - r_1)q_1(x)] = \deg(x - r_1) + \deg q_1$, we find $\deg q_1 = n - 1$. Now we apply the Fundamental Theorem to $q_1(x)$ to get

$$f(x) = (x - r_1)(x - r_2)q_2(x),$$

where $\deg q_2 = n - 2$. Thus, we can continue applying the Fundamental Theorem until finally we have the desired factorization

$$f(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_n).$$

Showing the roots exist is one thing; finding them is another thing altogether. Rather than provide a recipe-like formula, the best we can do is give a batch of methods to guide us to the roots.

For the *rational* roots of a polynomial, there is a method we can use to narrow the search. Although there are infinitely many rational numbers we could guess as roots of $f(x)$, the only ones which have a chance of being roots are given by the **Rational Root Theorem**. For any polynomial

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$$

with integer coefficients, all rational roots are of the form p/q, where $|p|$ and $|q|$ are relatively prime integers, p divides a_0 evenly, and q divides a_n evenly. The Rational Root Theorem will be proven as an example on page 58.

EXAMPLE 6-4 Find all the roots of $x^3 - 6x^2 + 11x - 6$.

Solution: From the Rational Root Theorem, we know that all possible roots are of the form p/q, where p divides -6 and q divides 1. Thus the possible roots are $\{\pm 1, \pm 2, \pm 3, \pm 6\}$. If we substitute these in the polynomial, we find that $\{1, 2, 3\}$ all satisfy $f(x) = 0$, so these are the three roots of the polynomial. (How do we know there aren't any more?)

EXAMPLE 6-5 Find all the roots of $2x^3 - 5x^2 + 4x - 1$.

Solution: Once again we apply the Rational Root Theorem and determine that the possible roots are $\{\pm 1/2, \pm 1\}$. Trying these, we find that both 1 and $1/2$ are roots of the polynomial. We know that there must be one more (why?), but we also know that no other rationals could possibly be roots. We might think that the third root is irrational or perhaps imaginary, but as we will see, no polynomial with rational coefficients can have just one irrational or one imaginary root. Thus, we come to the conclusion that this polynomial must have a double root, just like quadratic expressions which are perfect squares, such as $x^2 + 2x + 1$. Indeed, in this problem, we can use synthetic division to find $(2x^3 - 5x^2 + 4x - 1)/(x - 1) = 2x^2 - 3x + 1$. Factoring this quadratic, we find

$$2x^3 - 5x^2 + 4x - 1 = (x - 1)^2(2x - 1),$$

so that the root $x = 1$ is a **double root**, meaning the factor $(x - 1)$ occurs twice.

We have already come across two shortcomings of using the Rational Root Theorem alone. One is that we will miss multiple roots. Another is that it could still end up taking a very long time, as there are many numbers for polynomials like $12x^4 - x - 60$ which satisfy the Rational Root Theorem criteria.

To avoid missing multiple roots and to shorten our search for the roots, when we find a root r_1 of the polynomial, we divide $(x - r_1)$ into $f(x)$, as

$$f(x) = (x - r_1)q(x).$$

Then, we continue our search for roots with $q(x)$, because all roots of $q(x)$ are also roots of $f(x)$. As we saw in the previous section, synthetic division provides a swift method for performing the division.

EXAMPLE 6-6 Prove the Rational Root Theorem.

Proof: Let p/q be a rational root of the polynomial $f(x)$, where p and q are relatively prime positive integers. The case where the root is $-p/q$ is virtually the same. Since p/q is a root, we have

$$f\left(p/q\right) = a_n \left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_0 = 0.$$

Multiplying by q^n gives

$$a_n p^n + a_{n-1}p^{n-1}q + \cdots + a_1 pq^{n-1} + a_0 q^n = 0.$$

Now look at this equation modulo p. The first n terms on the left will become 0 since they are multiples of p, so we have

$$a_n p^n + a_{n-1}p^{n-1}q + \cdots + a_1 pq^{n-1} + a_0 q^n \equiv 0 + \cdots + 0 + a_0 q^n \pmod p \equiv 0 \pmod p$$

Thus, $a_0q^n \equiv 0 \pmod{p}$, so $p|a_0q^n$. Since p and q are relatively prime, it follows that $p|a_0$.

By the same argument, we can evaluate the sum mod q to show that $q|a_np^n$. Thus $q|a_n$ and the proof is complete.

There are a few more guides to tell us where to look for roots. The first is **Descartes' Rule of Signs,** which gives us a method to count how many positive and how many negative roots there are. We do this by counting sign changes. The number of sign changes in the coefficients of $f(x)$ (meaning we list the coefficients from first to last and count how many times they change from positive to negative) tells us the maximum number of positive roots the polynomial has, and the number of sign changes in the coefficients of $f(-x)$ gives us the maximum number of negative roots the polynomial has. Hence, for

$$f(x) = 3x^5 + 2x^4 - 3x^2 + 2x - 1,$$

there are at most 3 positive roots and at most 2 negative roots (since $f(-x) = -3x^5 + 2x^4 - 3x^2 - 2x - 1$). Furthermore, the actual number of positive or negative roots will always differ by an even number from the aforementioned maximum, so our above $f(x)$ has 1 or 3 positive roots and 0 or 2 negative roots.

Another root location method is finding upper and lower bounds. Suppose we use synthetic division to find $f(x)/(x - c)$ where $f(x)$ has a positive leading coefficient and $c \geq 0$ as below:

$$\begin{array}{r|rrrr} 3 & 1 & -1 & 2 & 6 \\ & & 3 & 6 & 24 \\ \hline & 1 & 2 & 8 & 30 \end{array}$$

If all the resulting coefficients in the quotient are positive (including the remainder), as in the example above, then no roots are greater than c. (Why?) This c is called an **upper bound** on the solutions since no roots can be higher. Similarly, if $c < 0$ and the coefficients of the quotient and remainder alternate in sign, then there is no root smaller than c (which we then call a **lower bound** for the roots). Locating upper and lower bounds will often help you shorten your search for roots.

Lastly, recall from our discussion of quadratic equations in Volume 1 that complex roots and roots of the form $a + b\sqrt{c}$ come in pairs if the coefficients of the quadratic are rational. This is also true of any polynomial with rational coefficients. For example, if the complex number $z = a + bi$ is a root of $f(x)$, we have

$$f(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 = 0.$$

Now we use some of our useful properties of complex numbers, such as $\overline{w+z} = \overline{w} + \overline{z}$, $\overline{z^k} = (\overline{z})^k$, and $w = z$ implies $\overline{w} = \overline{z}$. Applying these principles to $f(z)$, we have

$$\begin{aligned} \overline{a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0} &= \overline{0} \\ \overline{a_nz^n} + \overline{a_{n-1}z^{n-1}} + \cdots + \overline{a_1z} + \overline{a_0} &= 0 \\ a_n\overline{z}^n + a_{n-1}\overline{z}^{n-1} + \cdots + a_1\overline{z} + a_0 &= 0. \end{aligned}$$

Hence, if $f(z) = 0$, then $f(\overline{z}) = 0$, so $\overline{z}$ is also a root. This proof, with slight modifications, can be used to show that if $z = a + b\sqrt{c}$ is a root, then $z = a - b\sqrt{c}$ is also a root.