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Asymptotic enumeration of sparse strongly connected digraphs by vertices and edges

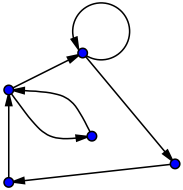
Xavier Pérez Giménez, Nicholas C. Wormald

Graphs @ Ryerson Seminar

Toronto, November, 2012

Strongly connected digraphs

How many?



n vertices

m arcs

$(n \leq m \leq n^2)$

Some old results:

- Moon, Moser '66: Almost all 2^{n^2} digraphs on *n* vertices are strongly connected (2-paths)
- Palásti '66: If $m = \lfloor n \log n + \alpha n \rfloor$ for fixed α, then $\Pr(\text{strongly connected}) \to \exp(-2e^{-\alpha})$.
- Wright '77: Recurrences for the exact number, for $m - n = O(1)$

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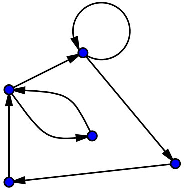

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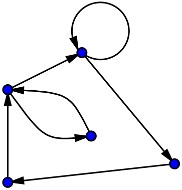

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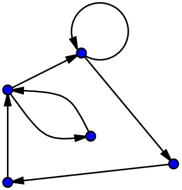

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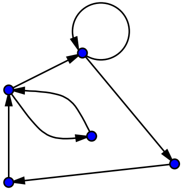

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Theorem (Cooper, Frieze '04):

Asymptotic formula for **Pr**(strongly connected) for a fixed 'nice' degree sequence and $(1+\epsilon)n \leq m = O(n)$.

Theorem (Pittel '12 / PG, Wormald '12):

The number of strongly connected digraphs is

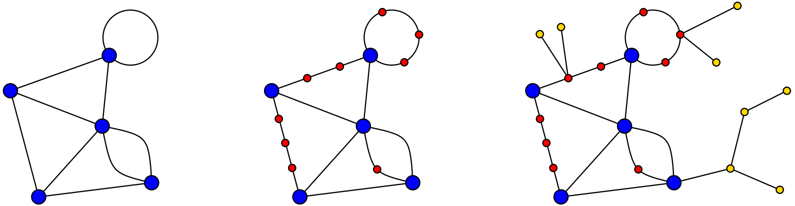
$$\sim \frac{(m-1)!(e^{\lambda}-1)^{2n}}{2\pi(1+\lambda-m/n)\lambda^{2m}} \exp(-\lambda^2/2) \frac{e^{\lambda}(e^{\lambda}-1-\lambda)^2}{(e^{2\lambda}-e^{\lambda}-\lambda)(e^{\lambda}-1)},$$

where λ is determined by $m/n = \lambda e^{\lambda}/(e^{\lambda}-1)$.

- For $m = O(n)$ and $m - n \gg n^{2/3}$. Explicit error estimates.
- For $m = O(n \log n)$ and $m - n \to \infty$. Also loop-free case.

Similar problems

- Bender, Canfield, McKay '90 / Pittel, Wormald '05 / van der Hofstad, Spencer '06: Number of connected graphs with n vertices and m edges
- Kemkes, Sato, Wormald '12: Number of 2-connected graphs with n vertices and m edges
- What about 3-connected graphs? Easy

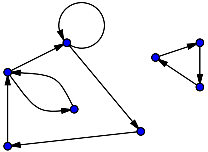


Dicores

(k^+, k^-)-dicore:

min outdegree $\geq k^+$; min indegree $\geq k^-$

dicore = $(1,1)$-dicore



Theorem (PG,Wormald '12):

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Extension to (k^+, k^-)-dicores for fixed $k^+, k^- \in \mathbb{Z}^+$.

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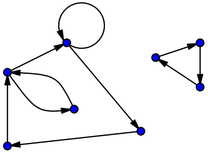

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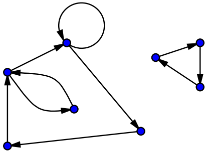

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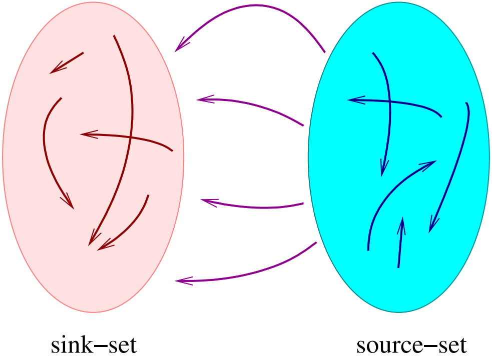
$\Longleftrightarrow$

no sink/source-sets



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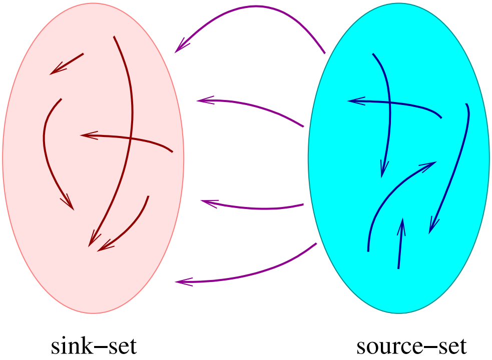

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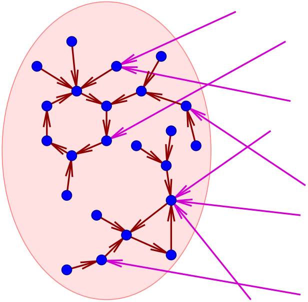

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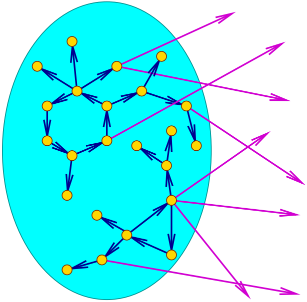

plain source–set

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$\begin{cases} \text{no sink/source-cycle \&} \\ \text{no complex sink/source-set of } \leq m/2 \text{ arcs} \end{cases}$

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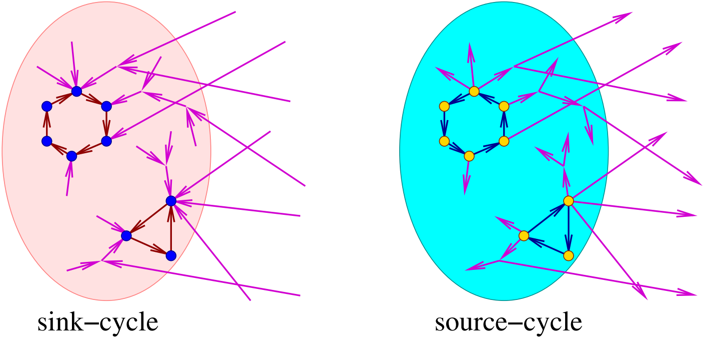

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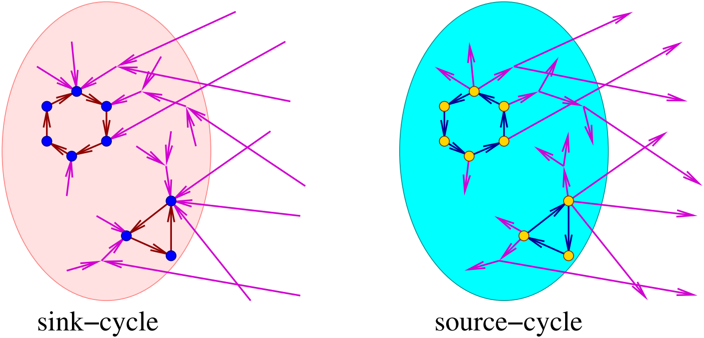

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Degree sequence and pairings

- Out-degree and in-degree sequences:

$$\vec{d} = (d^+, d^-), \quad d^+ = (d_1^+, \ldots, d_n^+), \quad d^- = (d_1^-, \ldots, d_n^-)$$

- Choose as independent truncated Poisson:

$$\mathbf{Pr}(Y = k) = \begin{cases} \frac{1}{e^\lambda - 1} \frac{\lambda^k}{k!} & \text{if } k \geq 1 \\ 0 & \text{if } k = 0 \end{cases}, \quad \mathbf{E}\,Y = \frac{\lambda e^\lambda}{e^\lambda - 1} = \frac{m}{n}$$

- Condition on sum: $\sum_{i=1}^{n} d_i^+ = \sum_{i=1}^{n} d_i^- = m$
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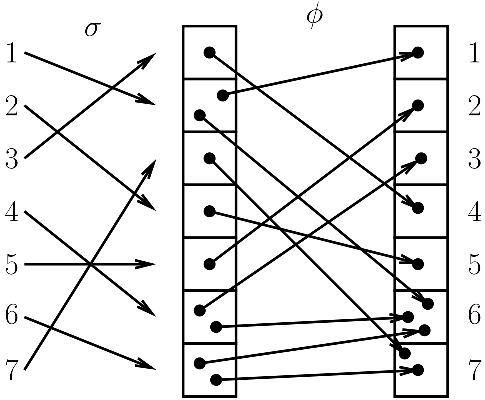

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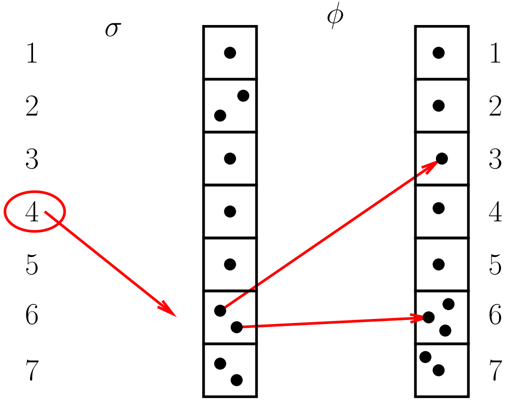

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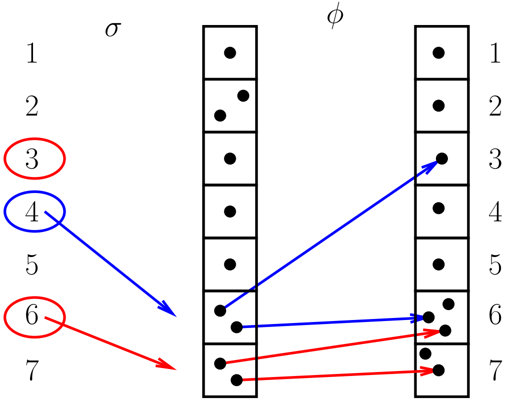

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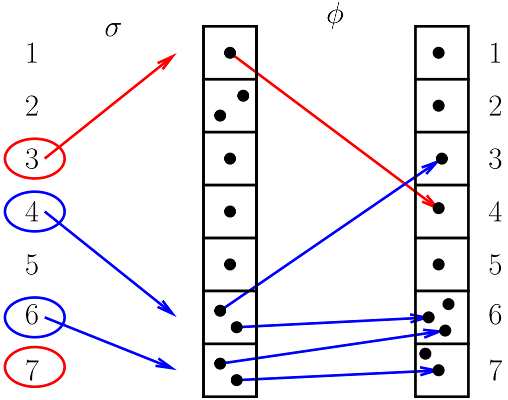

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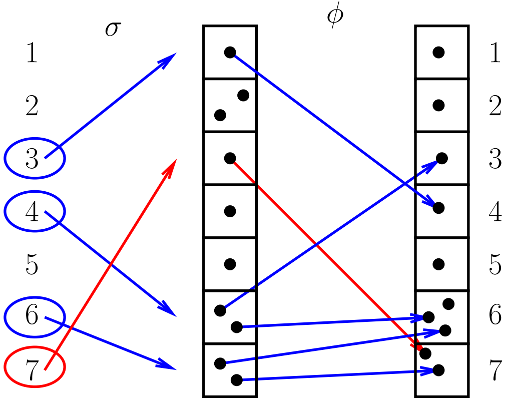

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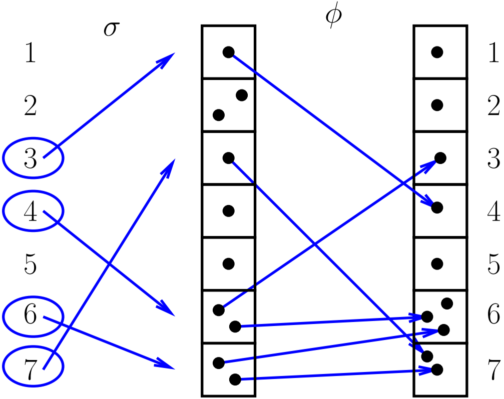

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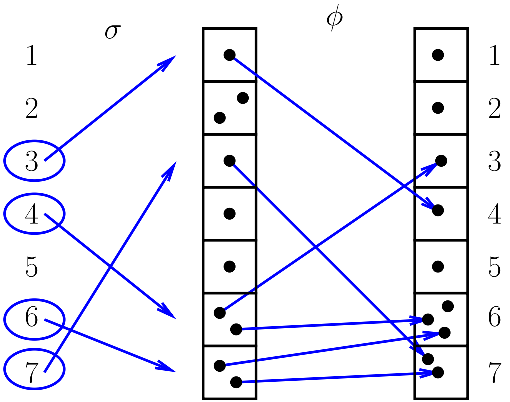

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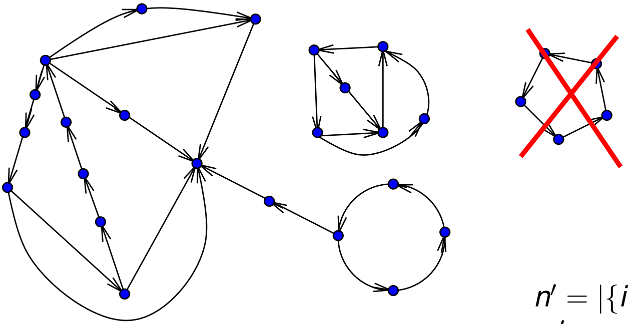

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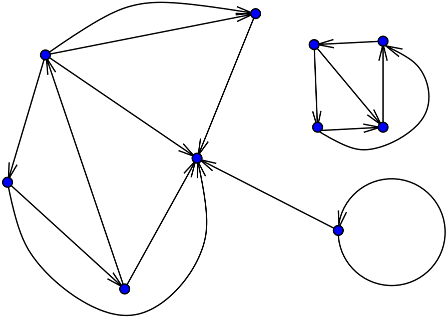

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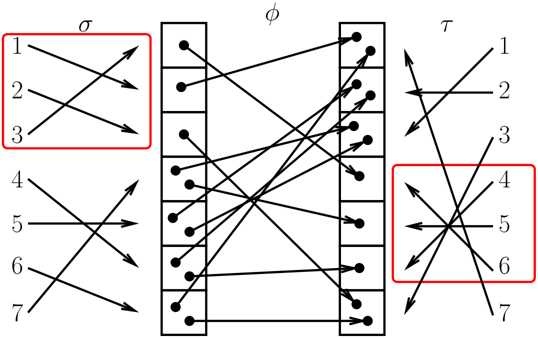

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A.a.s. no sink/source-sets of size $1 \leq s \leq n/2$.

Case $1 \leq s \leq (m/n)^K$ (switching argument):

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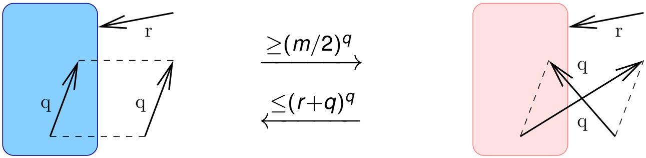

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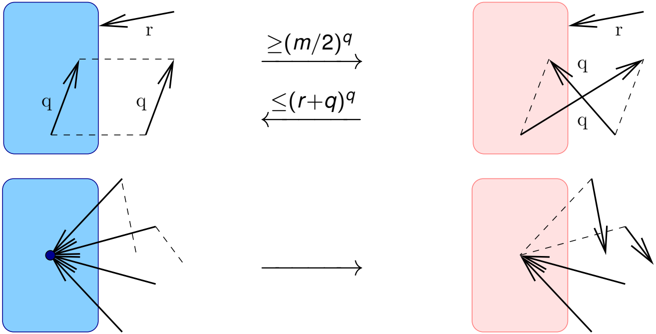

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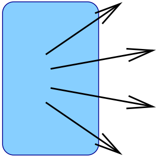

Further work

Study the structure of the strongly connected component in the supercritical phase of the evolution of the random digraph.

Thank you!