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The chromatic number of random regular graphs

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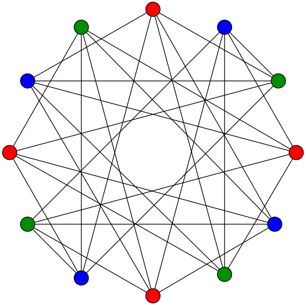
Models of random graphs

- Erdős & Rényi classic models ($\mathscr{G}_{n,p}$, $\mathscr{G}_{n,m}$)
- Fixed degree sequence
- Regular graphs ($\mathscr{G}_{n,d}$)
  - Drawn u.a.r. from the set of d-regular graphs on n vertices.
  - Asymptotic statements as $n \to \infty$. (Assume dn is even.)

The problem



Chromatic number:

$\chi(G) = 3$

What is $\chi(\mathscr{G}_{n,d})$?

Some results

Thm (Molloy & Reed 1992)

Let d be constant. If $q(1-1/q)^{d/2} < 1$, then a.a.s. $\chi(\mathscr{G}_{n,d}) > q$.

Thm (Frieze & Łuczak 1992)

Let $d < n^{1/3-\epsilon}$. Then a.a.s. $\chi(\mathscr{G}_{n,d}) = \frac{d}{2\log d} + O\left(\frac{d \log\log d}{\log^2 d}\right)$.

Thm (Cooper, Frieze, Reed & Riordan 2002)

Same for $d \leq n^{1-\epsilon}$.

Thm (Krivelevich, Sudakov, Vu & Wormald 2001)

Let $n^{6/7+\epsilon} \leq d \leq 0.9n$. Then a.a.s. $\chi(\mathscr{G}_{n,d}) \sim \frac{n}{2\log_{n/(n-d)} d}$.

Some results

Thm (Ben-Shimon & Krivelevich 2009)

For $d = o(n^{1/5})$, $\chi(\mathscr{G}_{n,d})$ is a.a.s. two-point concentrated.

Thm (Achlioptas & Moore 2004)

(For fixed $d \geq 3$.) Let k be the smallest integer s.t. $d < 2(k-1)\log(k-1)$. Then a.a.s. $\chi(\mathscr{G}_{n,d}) \in \{k-1, k, k+1\}$. If moreover $d > (2k-3)\log(k-1)$, then a.a.s. $\chi(\mathscr{G}_{n,d}) \in \{k, k+1\}$.

Thm (Shi & Wormald 2007)

A.a.s.

d	4	5	6	7, 8, 9	10
$\chi(\mathscr{G}_{n,d})$	3	$\{3,4\}$	4	$\{4,5\}$	$\{5,6\}$

Some results

Thm (Díaz, Kaporis, Kemkes, Kirousis, P.G. & Wormald 2009)

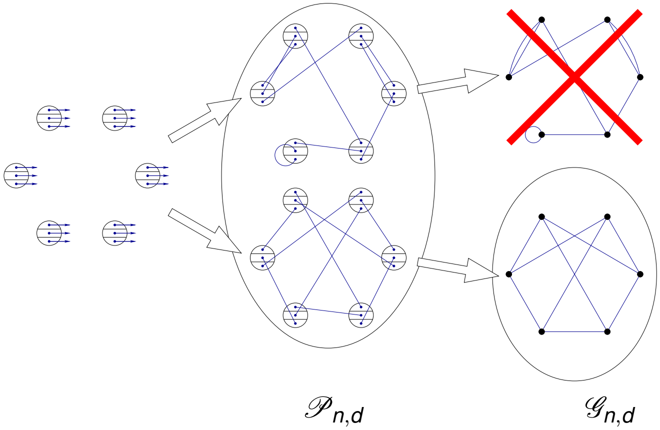
A.a.s. $\chi(\mathscr{G}_{n,5}) = 3 \ldots$ unless *‘there are kangaroos on the moon’*.

Thm (Kemkes, P.G. & Wormald 2010)

(For fixed $d \geq 3$.) Let k be the smallest integer s.t. $d < 2(k-1)\log(k-1)$. Then a.a.s. $\chi(\mathscr{G}_{n,d}) \in \{k-1, k\}$.
If moreover $d > 2(k-3)\log(k-1)$, then a.a.s. $\chi(\mathscr{G}_{n,d}) = k$.

So e.g. $\chi(\mathscr{G}_{n,10}) = 5$ and $\chi(\mathscr{G}_{n,10^6}) = 46523$.

The configuration model



The second moment method

Lemma

Let $X \geq 0$ be a discrete random variable (e.g. # of proper colourings). Then $\mathbf{P}(X > 0) \geq \dfrac{(\mathbf{E}X)^2}{\mathbf{E}X^2}$

Thus, if $\mathbf{E}X^2 \sim (\mathbf{E}X)^2$ then a.a.s. $X > 0$.

Problem!

$\mathbf{E}X^2 = \Theta\left((\mathbf{E}X)^2\right)$. Variance is too large!

This only shows that $X > 0$ with probability bounded away from 0.

The small subgraph conditioning method

Why??

Let $\boldsymbol{Z} = (Z_1, Z_2, Z_3, \ldots)$ be the cycle counts.
Law of total variance: $\mathbf{Var}X = \mathbf{E}(\mathbf{Var}(X|\boldsymbol{Z})) + \mathbf{Var}(\mathbf{E}(X|\boldsymbol{Z}))$
Unexplained / explained variance

Thm (Janson 1995; Robinson & Wormald 1992)

(i) Let $\lambda_1, \lambda_2, \ldots \geq 0$ and $\delta_1, \delta_2, \ldots \geq -1$ with $\sum_i \lambda_i {\delta_i}^2 < \infty$;
(ii) $Z_1, \ldots, Z_k$ asymptotically indep. Poisson with $\mathbf{E}Z_i \sim \lambda_i$;
(iii) $\mathbf{E}(X[Z_1]_{j_1} \cdots [Z_k]_{j_k} \sim \prod_{i=1}^{k} (\lambda_i(1+\delta_i))^{j_i}$; and
(iv) $\mathbf{E}X^2/(\mathbf{E}X)^2 \sim \exp\left(\sum_i \lambda_i {\delta_i}^2\right)$.

Then $\mathbf{P}(X > 0) \sim \exp\left(-\sum_{\delta_i=-1} \lambda_i\right)$ and [...] (contiguity)

The moments

- First moment:

$$\mathbf{E}X = \frac{\left|\left\{(G, C) \mid G \in \mathscr{P}_{n,d}, C \models G\right\}\right|}{\left|\mathscr{P}_{n,d}\right|}$$

- Second moment:

$$\mathbf{E}X^2 = \frac{\left|\left\{(G, C_1, C_2) \mid G \in \mathscr{P}_{n,d}, C_1, C_2 \models G\right\}\right|}{\left|\mathscr{P}_{n,d}\right|}$$

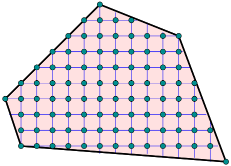
- Joint moments:

$$\mathbf{E}(X[Z_1]_{j_1} \cdots [Z_k]_{j_k}) = \frac{\left|\left\{(c_1, \ldots, c_k, G, C) \mid \cdots\right\}\right|}{\left|\mathscr{P}_{n,d}\right|}$$

Asymptotics

moment M

Polytope $\mathcal{D}$



$\dim(\mathcal{D}) = r$

$\mathcal{I} = \mathcal{D} \cap \left(\frac{1}{n}\mathbb{Z}\right)^r$

- Exact expression:
  $$M = \sum_{i \in \mathcal{I}} T_i$$
- Asymptotic expression:
  $$M \sim \sum_{i \in \mathcal{I}} \mathrm{poly}_i(n)(\widehat{F}_i)^n$$
- Exponential behaviour:

  If $\widehat{F}_{i_0} \geq \widehat{F}_i \ (\forall i \in \mathcal{I})$ then $M \asymp (\widehat{F}_{i_0})^n$.
  (Global non-linear optimisation)
- Polynomial factors:
  $$M = \Theta\left(n^{r/2}\right)\mathrm{poly}_{i_0}(n)(\widehat{F}_{i_0})^n$$
  (Laplace integration technique/ Saddlepoint method)

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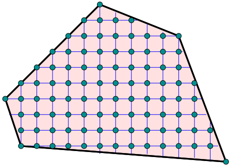

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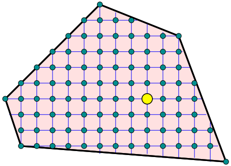

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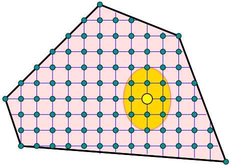

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What about $d = 5$?

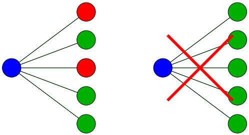
We know $\chi(\mathscr{G}_{n,5}) \in \{3, 4\}$. *Is it 3?*

Problem!

If X = # of proper 3-colourings, then $\mathbf{E}X^2 = \omega((\mathbf{E}X)^2)$.
So the second moment method is useless.

Idea: Balanced Rainbow 3-Colourings

- **Balanced:** $n/3$ vertices of each colour.
- **Rainbow:** No single vertex can legally change colour. (No vertex has mono-chromatic neighbors).



It works

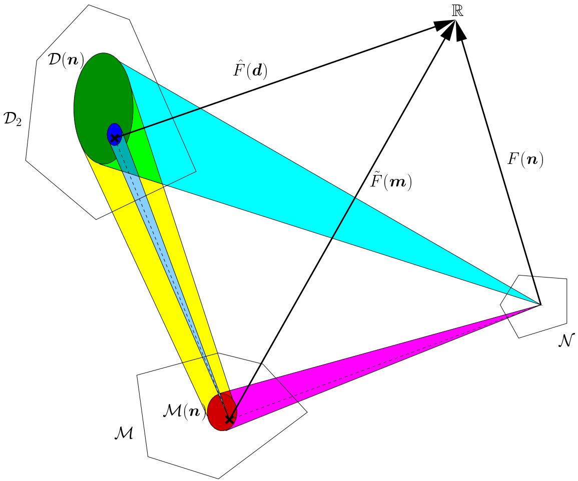
Thm

- $\mathbf{E}X \sim \sqrt{\frac{2^2 3^6 5^3}{11^3} \frac{1}{(2\pi n)^2} \left(\frac{25}{24}\right)^n}$
- If no kangaroos… $\mathbf{E}X^2 \sim \frac{2^2 3^{19} 5^{16}}{7^6 11^7 79^2 \sqrt{13\,17}} \frac{1}{(2\pi n)^2} \left(\frac{25}{24}\right)^n$
- All other required conditions for the small subgraph conditioning method are satisfied.

Kangaroos on the moon?

- Maximisation of $\widehat{F}$ in a domain $\mathcal{D}$, $\dim(\mathcal{D}) = 301$.
  We have a local maximum candidate. Is it the global too?
- We transform the problem *(Lagrange multipliers)*
  - $\max_{\mathcal{D}}(\widehat{F}) \Longleftrightarrow \max_{\mathcal{N}}(F)$
  - **Good:** $\dim \mathcal{N} = 4$.
  - **Bad:** F is implicit.
- We find $\max_{\mathcal{N}}(F)$ numerically
  - Fine sweep of the domain (step $= 10^{-4}$).
  - IBM-Mare Nostrum (Barcelona Supercomputing Center):
    2.268 dual 64-bit processor blade nodes,
    4.536 2.2 GHz PPC970FX processors.

Maximisation of $\widehat{F}$



- $\mathcal{D} \subset \mathbb{R}^{324}$, dim $\mathcal{D} = 301$
- $\mathcal{M} \subset \mathbb{R}^{36}$, dim $\mathcal{M} = 13$
- $\mathcal{N} \subset \mathbb{R}^{9}$, dim $\mathcal{N} = 4$
- $\mathcal{M}(\boldsymbol{n}) \subset \mathbb{R}^{36}$, dim $\mathcal{M}(\boldsymbol{n}) = 9$
- log $\widetilde{F}$ concave in $\mathcal{M}(\boldsymbol{n})$

Open Issues

- Case $d = 5$. Are there kangaroos on the moon?
- Close the gap for $\chi(\mathscr{G}_{n,d})$ for general constant d. Can we extend the idea of rainbow balanced colourings?
- Extend results to some $d \to \infty$.

Thank you!