Strong-majority bootstrap percolation on regular graphs

Xavier Pérez-Giménez[†]

joint work with

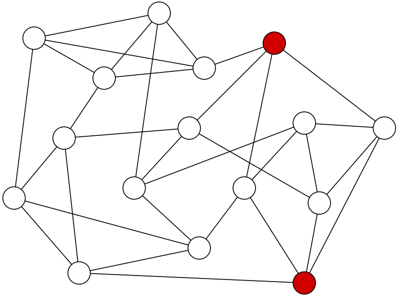
Dieter Mitsche[*] and Paweł Prałat[†]

[†]Ryerson University

[*]Université de Nice Sophia-Antipolis

Random Structures & Algorithms, CMU Pittsburgh, July 2015

Motivational example



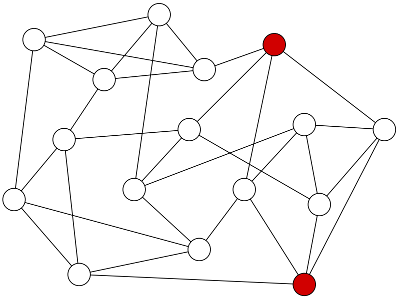
(Chalupa, Leath, Reich 1979)

Bootstrap percolation:

Given a connected graph,

- Pick initial active vertices (at random with prob. p).

Motivational example



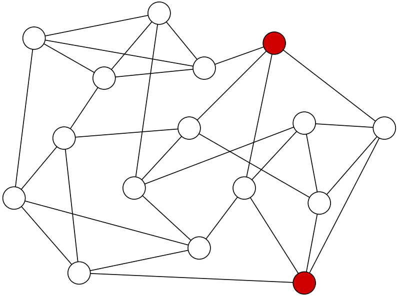
(Chalupa, Leath, Reich 1979)

Bootstrap percolation:

Given a connected graph,

- Pick initial active vertices (at random with prob. p).
- Active vertices stay active.

Motivational example



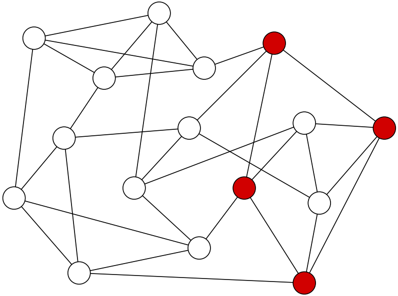
(Chalupa, Leath, Reich 1979)

Bootstrap percolation:

Given a connected graph,

- Pick initial active vertices (at random with prob. p).
- Active vertices stay active.
- **Rule:** inactive vertices with 2 active neighbours become active.

Motivational example



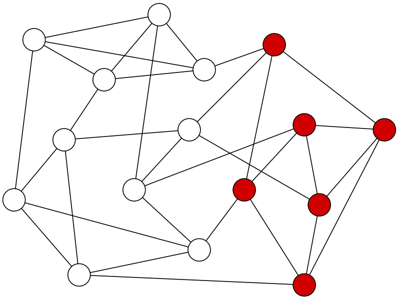
(Chalupa, Leath, Reich 1979)

Bootstrap percolation:

Given a connected graph,

- Pick initial active vertices (at random with prob. p).
- Active vertices stay active.
- **Rule:** inactive vertices with 2 active neighbours become active.

Motivational example



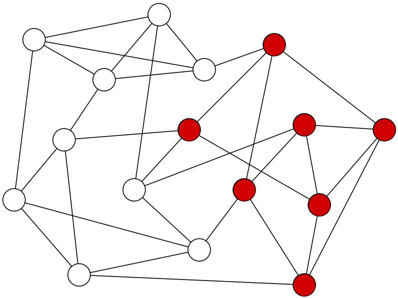
(Chalupa, Leath, Reich 1979)

Bootstrap percolation:

Given a connected graph,

- Pick initial active vertices (at random with prob. p).
- Active vertices stay active.
- **Rule:** inactive vertices with 2 active neighbours become active.

Motivational example

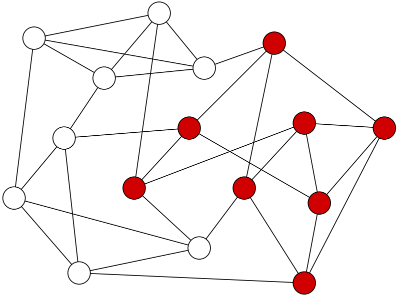


(Chalupa, Leath, Reich 1979)

Bootstrap percolation:

Given a connected graph,

- Pick initial active vertices (at random with prob. p).
- Active vertices stay active.
- **Rule:** inactive vertices with 2 active neighbours become active.



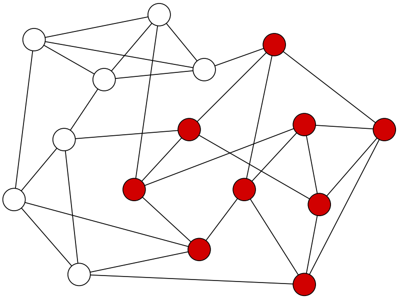
(Chalupa, Leath, Reich 1979)

Bootstrap percolation:

Given a connected graph,

- Pick initial active vertices (at random with prob. p).
- Active vertices stay active.
- **Rule:** inactive vertices with 2 active neighbours become active.

Motivational example



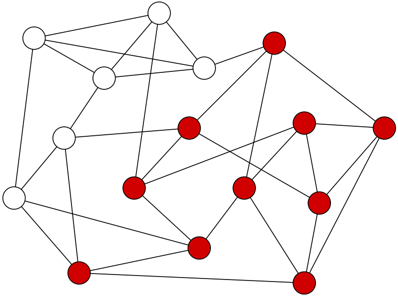
(Chalupa, Leath, Reich 1979)

Bootstrap percolation:

Given a connected graph,

- Pick initial active vertices (at random with prob. p).
- Active vertices stay active.
- **Rule:** inactive vertices with 2 active neighbours become active.

Motivational example



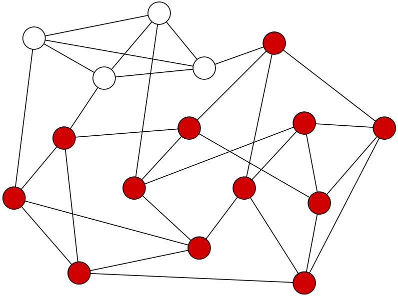
(Chalupa, Leath, Reich 1979)

Bootstrap percolation:

Given a connected graph,

- Pick initial active vertices (at random with prob. p).
- Active vertices stay active.
- **Rule:** inactive vertices with 2 active neighbours become active.

Motivational example

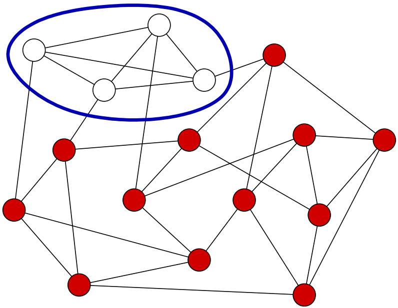


(Chalupa, Leath, Reich 1979)

Bootstrap percolation:

Given a connected graph,

- Pick initial active vertices (at random with prob. p).
- Active vertices stay active.
- **Rule:** inactive vertices with 2 active neighbours become active.
- Do all vertices become active? (p-dissemination)



(Chalupa, Leath, Reich 1979)

Bootstrap percolation:

Given a connected graph,

- Pick initial active vertices (at random with prob. p).
- Active vertices stay active.
- **Rule:** inactive vertices with 2 active neighbours become active.
- Do all vertices become active? (p-*dissemination*)
- No (*inactive community*).

Models

Our model:

- Pick initial active vertices (at random with prob. p).
- Active vertices stay active.
- ***r*-Majority Rule:** inactive vertices with r more active than inactive neighbours become active. (For this talk, $r = 1$.)
- Goal: all vertices become active (p-dissemination)

Models

Our model:

- Pick initial active vertices (at random with prob. p).
- Active vertices stay active.
- ***r*-Majority Rule:** inactive vertices with r more active than inactive neighbours become active. (For this talk, $r = 1$.)
- Goal: all vertices become active (p-dissemination)

Variations:

- **Rules:** at least t active neighbours, (strict) majority rule, probabilistic rules...
- **Goal:** full/partial dissemination, fast dissemination.
- Choice of **initial** set of **active vertices:** random, deterministic.
- **Graph:** deterministic, random, real world...
- **Other:** possibility of going back to inactive, more than two possible states (cellular automaton).

Event A_n holds a.a.s. (asymptotically almost surely) if $\lim_{n\to\infty} \mathbf{P}(A_n) = 1$.

Critical probability:

$(G_n)_{n\in\mathbb{N}}$ sequence of graphs of increasing order.

- $p_c^+(G_n) = \inf\{p \in [0,1] : G_n \text{ } p\text{-disseminates a.a.s.}\}$
- $p_c^-(G_n) = \sup\{p \in [0,1] : G_n \text{ does not } p\text{-disseminate a.a.s.}\}$
- $p_c^-(G_n) \leq p_c^+(G_n)$. If equal, call it $p_c(G_n)$.

So

- If $p \leq (1-\epsilon)p_c(G_n)$ then a.a.s. G_n does not p-disseminate;
- If $p \geq (1+\epsilon)p_c(G_n)$ then a.a.s. G_n p-disseminates.

Some examples

Trivial example:

Let K_n be the complete graph on n vertices.
Then, $p_c(K_n) = 1/2$.
Proof idea: a.a.s. all vertices have $(1 + o(1))pn$ active neighbours.

Theorem (Balogh, Bollobás, Morris 2009):

Let Q_n be the n-th dimensional hypercube (2^n vertices).
Then, $p_c(Q_n) = 1/2$.

Our goal:

Find G_n with small $p_c(G_n)$ for the (strict) 1-majority model.

Strict vs. nonstrict majority

Let $[n]^d$ be the d dimensional grid on n^d vertices.

Theorem (Balogh, Bollobás, Duminil-Copin, Morris 2012):

For the 0-majority model, $p_c([n]^d) = 0$.

Strict vs. nonstrict majority

Let $[n]^d$ be the d dimensional grid on n^d vertices.

Theorem (Balogh, Bollobás, Duminil-Copin, Morris 2012):

For the 0-majority model, $p_c([n]^d) = 0$.

However:

For the 1-majority model, $p_c([n]^d) = 1$.

Strict vs. nonstrict majority

Let $[n]^d$ be the d dimensional grid on n^d vertices.

Theorem (Balogh, Bollobás, Duminil-Copin, Morris 2012):

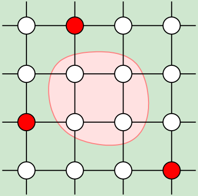
For the 0-majority model, $p_c([n]^d) = 0$.

However:

For the 1-majority model, $p_c([n]^d) = 1$.

Proof:

If $p < 1$, a.a.s. there are $\Theta(n^d)$ initially inactive d-cubes.



Strict vs. nonstrict majority

Let $[n]^d$ be the d dimensional grid on n^d vertices.

Theorem (Balogh, Bollobás, Duminil-Copin, Morris 2012):

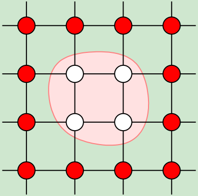
For the 0-majority model, $p_c([n]^d) = 0$.

However:

For the 1-majority model, $p_c([n]^d) = 1$.

Proof:

If $p < 1$, a.a.s. there are $\Theta(n^d)$ initially inactive d-cubes.



(Back to the (strict) 1-majority model...)

Theorem (Balogh, Pittel 2007):

Let $G_{d,n}$ be a random d-regular graph on n vertices.
Then, for $d \geq 3$ constant, $p_c(G_{d,n}) = p_d$.

d	3	4	5	6	7	8	9	
p_d	0.5	0.667	0.275	0.397	0.269	0.354	0.275	...

$\lim_{d\to\infty} p_d = 0.5$; $\min\{p_d : d \geq 3\} = p_7 \approx 0.269$.

Related work

Let $H_{d,n}$ denote a sequence of d-regular graphs of increasing order.

Rapaport, Suchan, Todinca, Verstraete 2011:

Theorem:

For any $H_{3,n}$, $\quad p_c^-(H_{3,n}) \geq p_3 = 1/2$.

Conjecture:

For all $d \geq 3$ and any $H_{d,n}$, $\quad p_c^-(H_{d,n}) \geq p_d$.

Theorem:

For any sequence $H_{d,n}$, $\quad p_c^-(H_{d,n}) \geq \begin{cases} 1/d & d \text{ odd}, \\ 2/d & d \text{ even}. \end{cases}$

Question:

Is there G_n s.t. $p_c(G_n) = 0$?

Our results

Theorem (Mitsche, P-G, Prałat 2015):

Suppose $10^6 \frac{(\log\log n)^{2/3}}{\log^{1/3} n} \le p(n) \le p_0$.
Define $k = \left\lfloor \frac{1000}{p} \log(1/p) \right\rfloor$.
Then, $L_{k,n}$ p-disseminates a.a.s.

$L_{k,n}$ is a (certain type of) $d = (4k+3)$-regular graph on n^2 vertices.

Corollary:

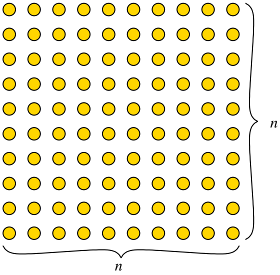
For any constant $0 < p < 1$, there is a d-regular sequence $H_{d,n}$ with $d = \Theta\left(\frac{1}{p}\log(1/p)\right)$ s.t. $p_c^+(H_{d,n}) \le p$.

Corollary:

If $k \to \infty$, then $p_c(L_{k,n}) = 0$.

Our graph

(Assume $p > 0$ small constant. Pick $k = k(p) \in \mathbb{N}$ large enough.)



Graph $\widetilde{L}_{k,n}$:

- $n \times n$ lattice on a torus.
- degree $4k + 2$.
- Activation rule: $\geq 2k + 2$ active neighbours.

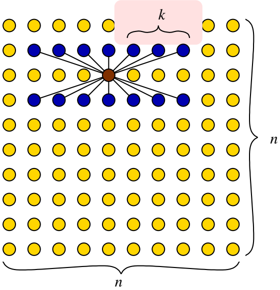
Graph $L_{k,n}$:

(Assume n even.)

- $\widetilde{L}_{k,n}$ + add random perfect matching (with no multiple edges)
- degree $4k + 3$
- Activation rule: $\geq 2k + 2$ active neighbours.

Our graph

(Assume $p > 0$ small constant. Pick $k = k(p) \in \mathbb{N}$ large enough.)



Graph $\widetilde{L}_{k,n}$:

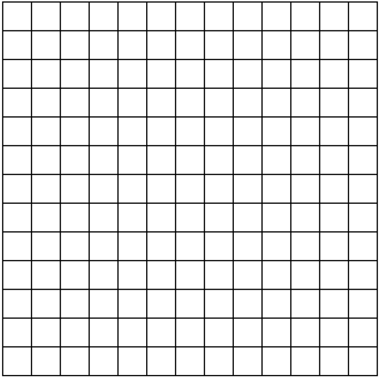
- $n \times n$ lattice on a torus.
- degree $4k + 2$.
- Activation rule: $\geq 2k + 2$ active neighbours.

Graph $L_{k,n}$:

(Assume n even.)

- $\widetilde{L}_{k,n}$ + add random perfect matching (with no multiple edges)
- degree $4k + 3$
- Activation rule: $\geq 2k + 2$ active neighbours.

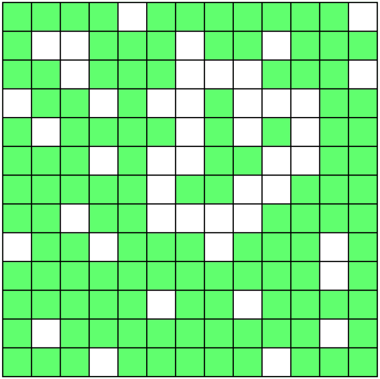
Proof strategy



Consider only $\widetilde{L}_{k,n}$:

- Tessellate torus into square cells of size $t \times t$.

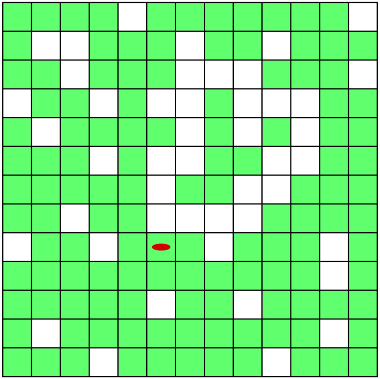
Proof strategy



Consider only $\widetilde{L}_{k,n}$:

- Tessellate torus into square cells of size $t \times t$.
- A.a.s. "most" cells are good. 

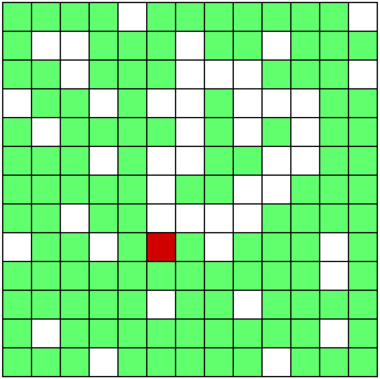
Proof strategy



Consider only $\widetilde{L}_{k,n}$:

- Tessellate torus into square cells of size $t \times t$.
- A.a.s. "most" cells are good.
- A.a.s. there is a seed.

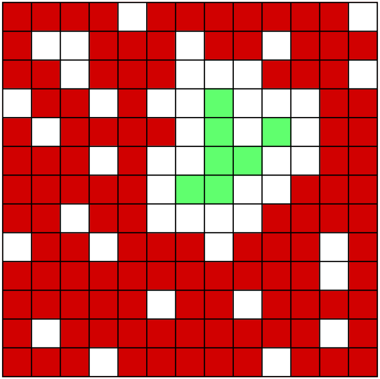
Proof strategy



Consider only $\widetilde{L}_{k,n}$:

- Tessellate torus into square cells of size $t \times t$.
- A.a.s. "most" cells are good.
- A.a.s. there is a seed.
- Deterministic spread over good cells. $\rightarrow$

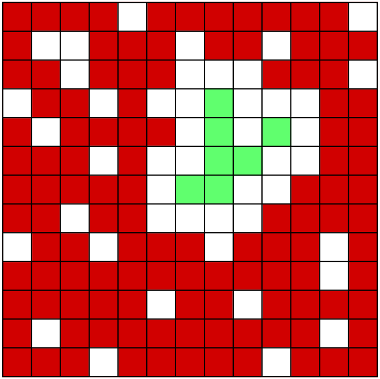
Proof strategy



Consider only $\widetilde{L}_{k,n}$:

- Tessellate torus into square cells of size $t \times t$.
- A.a.s. "most" cells are good.
- A.a.s. there is a seed.
- Deterministic spread over good cells. $\rightarrow$

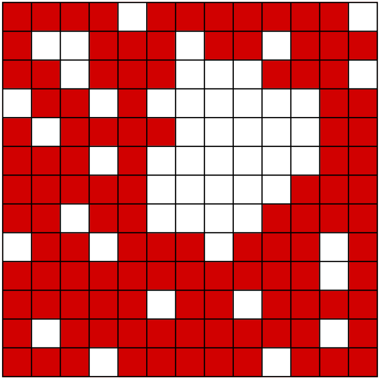
Proof strategy



Consider only $\widetilde{L}_{k,n}$:

- Tessellate torus into square cells of size $t \times t$.
- A.a.s. "most" cells are good.
- A.a.s. there is a seed.
- Deterministic spread over good cells. $\rightarrow$

Add random perfect matching:

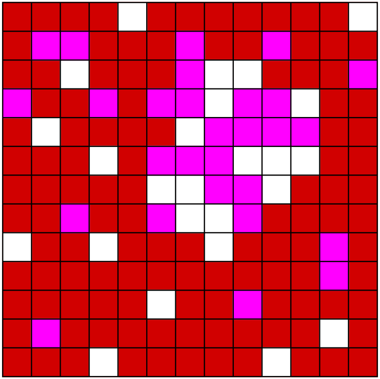


Consider only $\widetilde{L}_{k,n}$:

- Tessellate torus into square cells of size $t \times t$.
- A.a.s. "most" cells are good.
- A.a.s. there is a seed.
- Deterministic spread over good cells. $\rightarrow$

Add random perfect matching:

Proof strategy



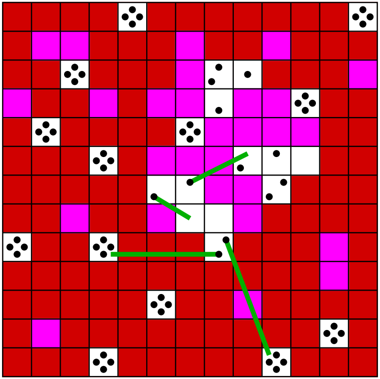
Consider only $\widetilde{L}_{k,n}$:

- Tessellate torus into square cells of size $t \times t$.
- A.a.s. "most" cells are good.
- A.a.s. there is a seed.
- Deterministic spread over good cells. $\rightarrow$

Add random perfect matching:

- Suppose there is inactive community.

Proof strategy

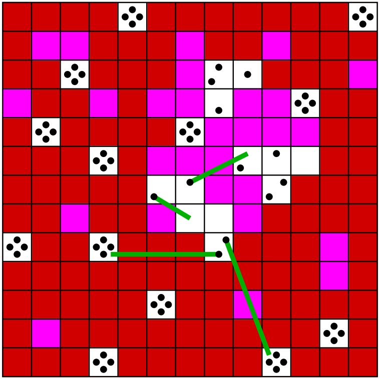


Consider only $\widetilde{L}_{k,n}$:

- Tessellate torus into square cells of size $t \times t$.
- A.a.s. "most" cells are good.
- A.a.s. there is a seed.
- Deterministic spread over good cells. $\rightarrow$

Add random perfect matching:

- Suppose there is inactive community.
- Necessary conditions.



Consider only $\widetilde{L}_{k,n}$:

- Tessellate torus into square cells of size $t \times t$.
- A.a.s. "most" cells are good.
- A.a.s. there is a seed.
- Deterministic spread over good cells. $\rightarrow$

Add random perfect matching:

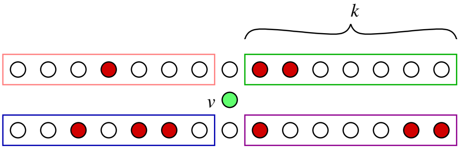
- Suppose there is inactive community.
- Necessary conditions.
- A.a.s. are not satisfied.

Good vertices

Definition

A vertex is good if

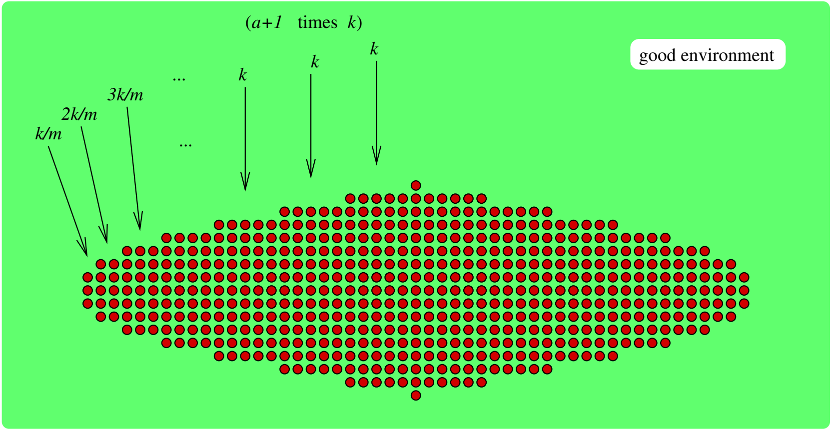
- it is active; or
- it has at least $3k/m$ active neighbours on the top-right, bottom-right, top-left and bottom-left neighbourhoods.



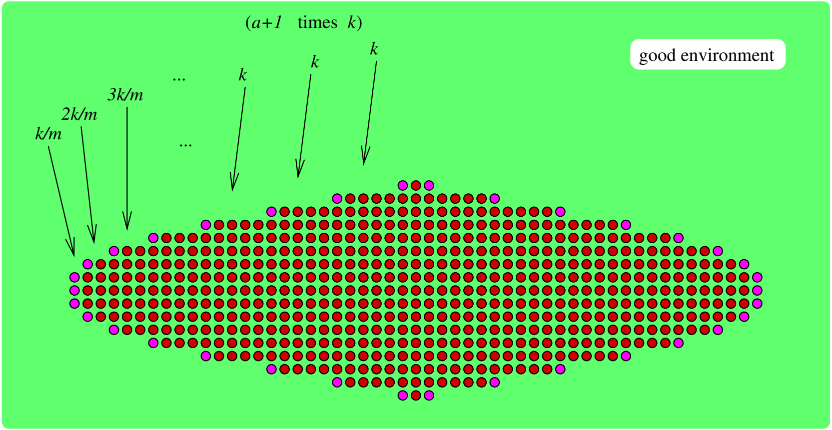
Lemma

A.a.s. “most” vertices are good.

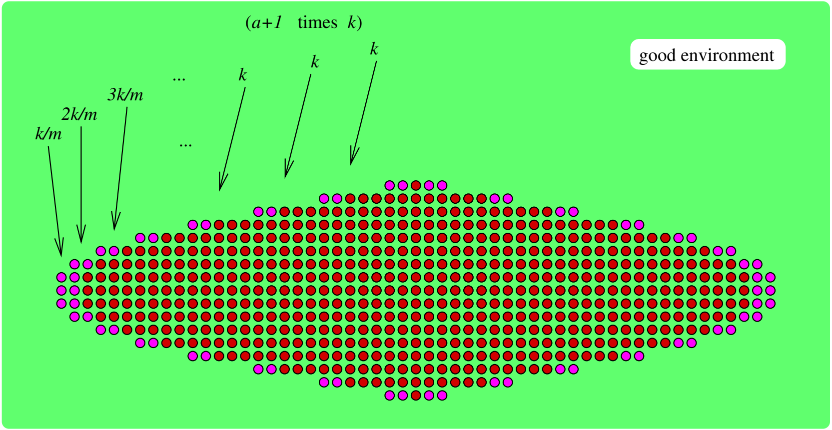
Deterministic growth



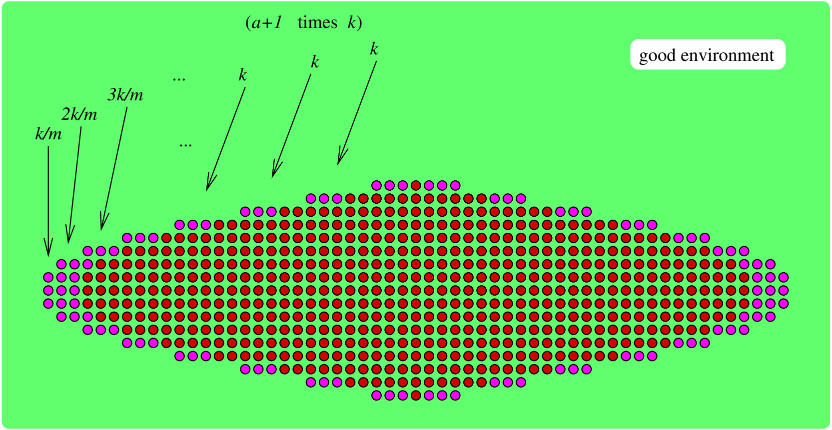
Deterministic growth



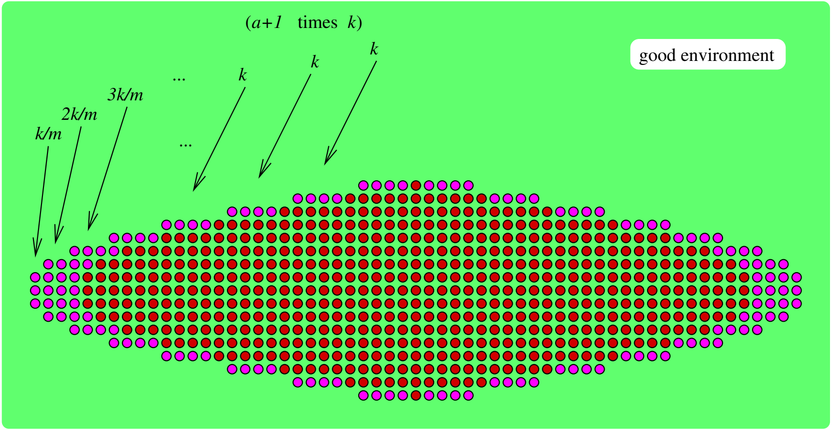
Deterministic growth



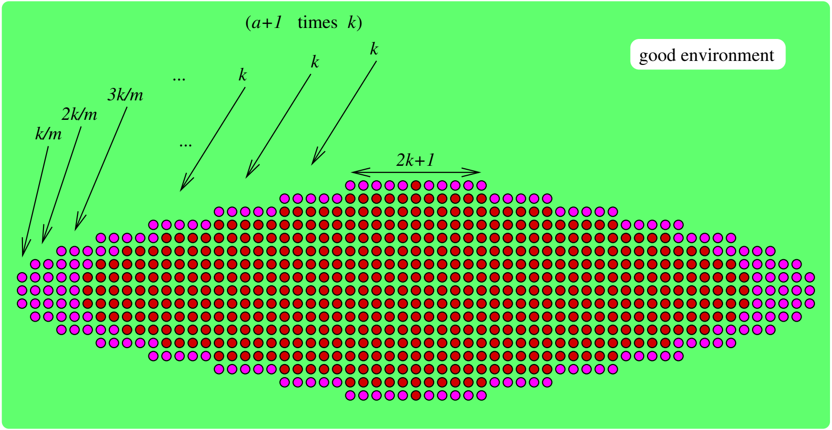
Deterministic growth



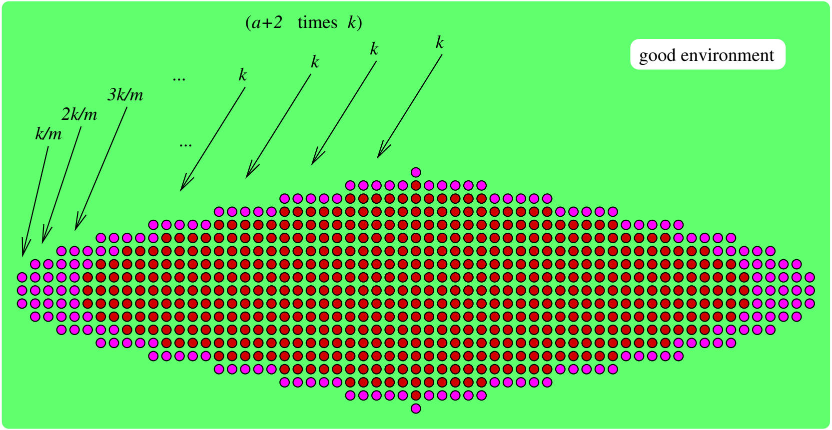
Deterministic growth



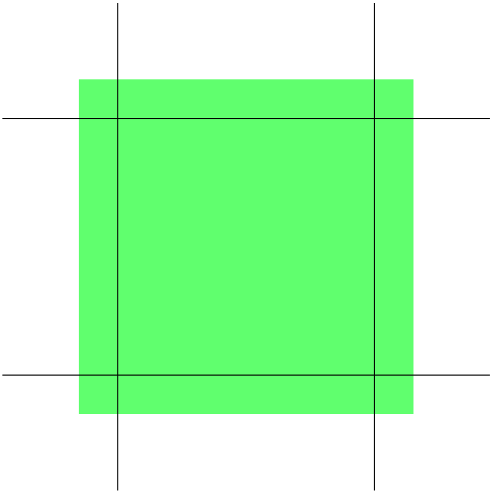
Deterministic growth



Deterministic growth

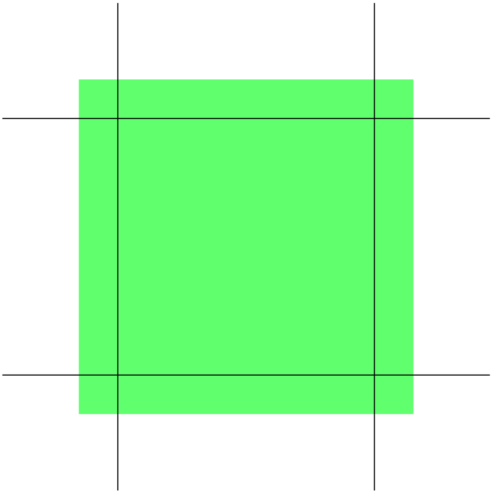


Good cells



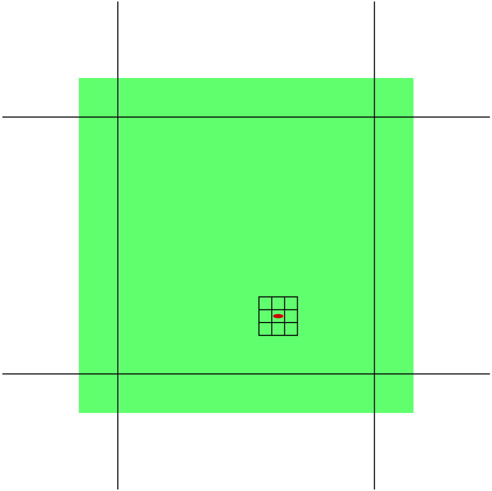
- Good cell: all vertices inside or “close” to it are good.

Good cells



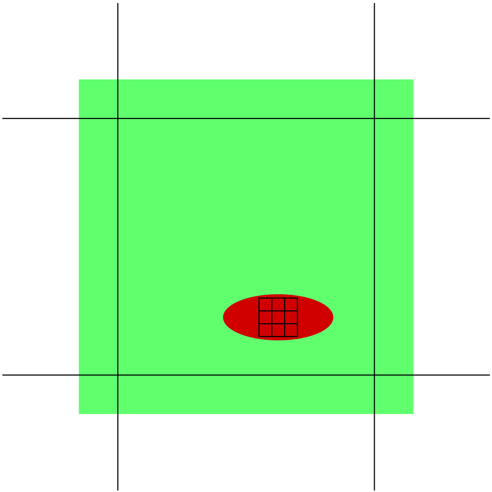
- Good cell: all vertices inside or “close” to it are good.
- Warning! We don’t have full independence.

Good cells



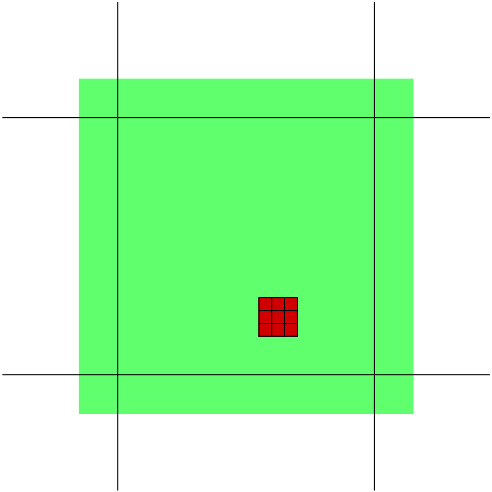
- Good cell: all vertices inside or "close" to it are good.
- Warning! We don't have full independence.

Good cells



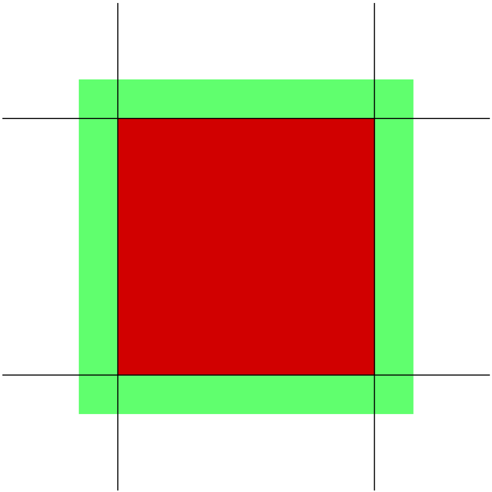
- Good cell: all vertices inside or "close" to it are good.
- Warning! We don't have full independence.

Good cells



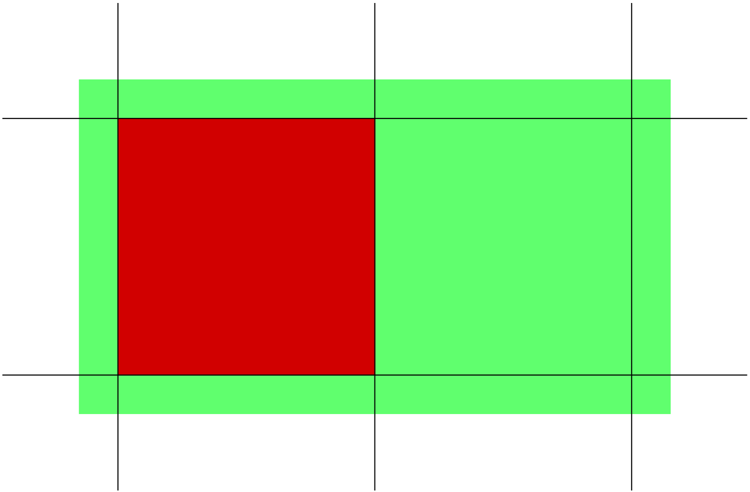
- Good cell: all vertices inside or "close" to it are good.
- Warning! We don't have full independence.

Good cells



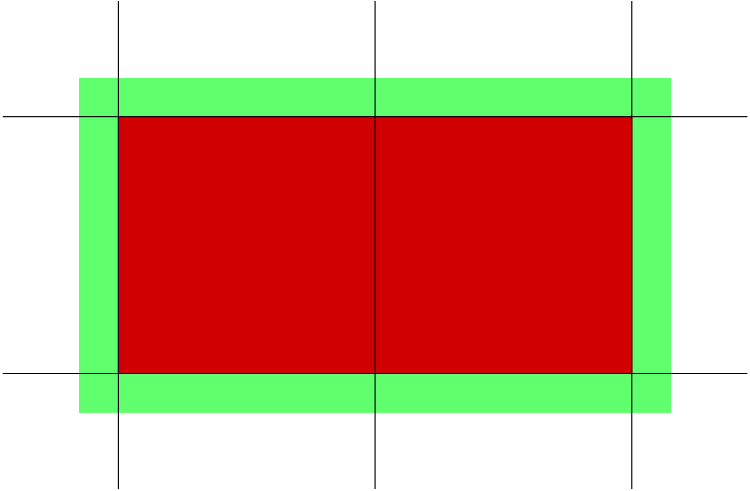
- Good cell: all vertices inside or "close" to it are good.
- Warning! We don't have full independence.

Good cells

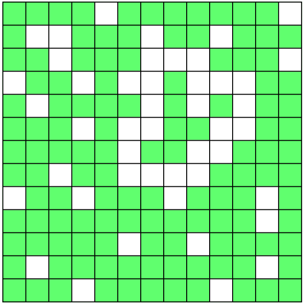


- Good cell: all vertices inside or “close” to it are good.
- Warning! We don’t have full independence.

Good cells



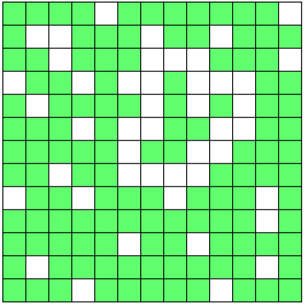
- Good cell: all vertices inside or “close” to it are good.
- Warning! We don’t have full independence.



Proposition:

Let Z be the largest "connected" set of good cells.

Percolative ingredients

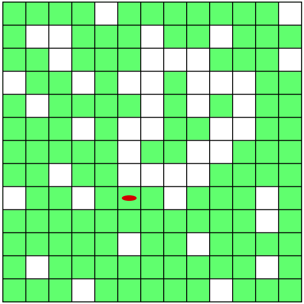


Proposition:

Let Z be the largest "connected" set of good cells.

- A.a.s. $|Z|$ is very large and "spread".

Percolative ingredients

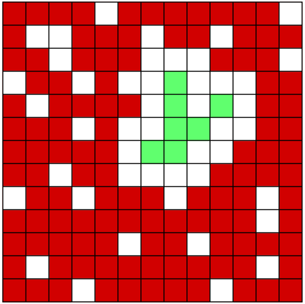


Proposition:

Let Z be the largest "connected" set of good cells.

- A.a.s. $|Z|$ is very large and "spread".
- A.a.s. Z contains an active seed.

Percolative ingredients

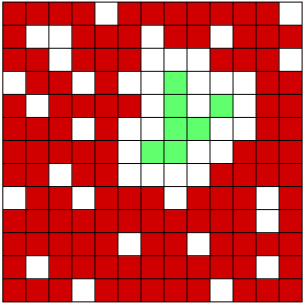


Proposition:

Let Z be the largest "connected" set of good cells.

- A.a.s. $|Z|$ is very large and "spread".
- A.a.s. Z contains an active seed.
- A.a.s. Z^C has only "small" connected pieces.

Percolative ingredients



Proposition:

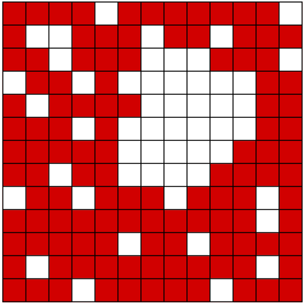
Let Z be the largest "connected" set of good cells.

- A.a.s. $|Z|$ is very large and "spread".
- A.a.s. Z contains an active seed.
- A.a.s. Z^C has only "small" connected pieces.

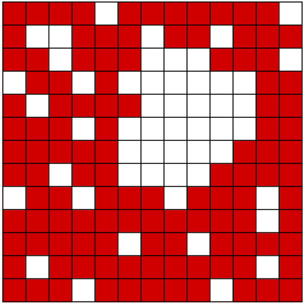
Proof technique:

The set of good cells behaves like a 2-dependent percolation model.

Suppose all previous events (about good cells) hold
Add a perfect matching M.



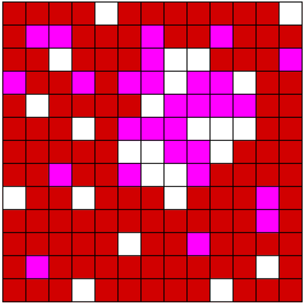
Suppose all previous events (about good cells) hold
Add a perfect matching M.



Lemma:

If some inactive vertices survive, then (deterministically):

Suppose all previous events (about good cells) hold
Add a perfect matching M.

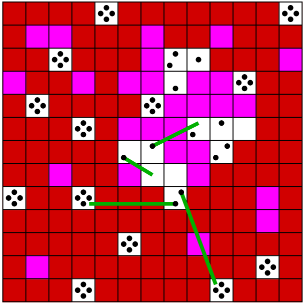


Lemma:

If some inactive vertices survive, then (deterministically):

- Innactive communities belong to small connected sets of cells.

Suppose all previous events (about good cells) hold
Add a perfect matching M.

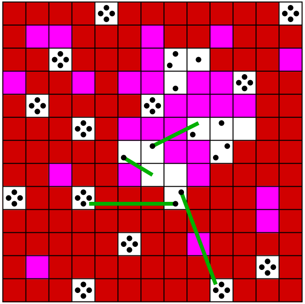


Lemma:

If some inactive vertices survive, then (deterministically):

- Innactive communities belong to small connected sets of cells.
- Each community has at least 4 vertices that must be matched to inactive vertices by M.

Suppose all previous events (about good cells) hold
Add a perfect matching M.



Lemma:

If some inactive vertices survive, then (deterministically):

- Innactive communities belong to small connected sets of cells.
- Each community has at least 4 vertices that must be matched to inactive vertices by M.

Lemma:

A.a.s. a random perfect matching M cannot satisfy the above.

Further strengthening

What about...

- even degree?
- stronger majority? (r-majority rule)

What about. . .

- even degree?
- stronger majority? (r-majority rule)

Answer:

Just add more perfect matchings!

Thank you