# Math 4/896: Seminar in Mathematics Topic: Inverse Theory

Instructor: Thomas Shores Department of Mathematics

Lecture 22, April 4, 2006 AvH 10

#### Total Variation

#### Key Property:

 TV doesn't smooth discontinuities as much as Tikhonov regularization.

Change startupfile path to Examples/chap7/examp3 execute it and examp.

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### Outline

### Problem is $G(\mathbf{m}) = \mathbf{d}$ with least squares solution $\mathbf{m}^*$ :

Now what? What statistics can we bring to bear on the problem?

- We minimize  $\|\mathbf{F}(\mathbf{m})\|^2 = \sum_{i=1}^{n} \frac{(G(\mathbf{m}) d_i)^2}{\sigma^2}$
- Treat the linear model as locally accurate, so misfit is  $\nabla \mathsf{F} = \mathsf{F} \left( \mathsf{m} + \Delta \mathsf{m} \right) - \mathsf{F} \left( \mathsf{m}^* \right) pprox \nabla \mathsf{F} \left( \mathsf{m}^* \right) \nabla \mathsf{m}$
- If  $\sigma$  is unknown but constant across measurements, take

$$s^{2} = \frac{1}{m-n} \sum_{i=1}^{m} (G(\mathbf{m}) - d_{i})^{2}$$

• Do confidence intervals,  $\chi^2$  statistic and p-value as in Chapter

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- Obtain covariance matrix  $Cov(\mathbf{m}^*) = \left(\nabla \mathbf{F}(\mathbf{m}^*)^T \nabla \mathbf{F}(\mathbf{m}^*)\right)^{-1}$
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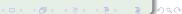
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### Outline

- Problem may have many local minima.
- Even if it has a unique solution, it might lie in a long flat basin.
- Analytical derivatives may not be available. This presents an interesting regularization issue not discussed by the authors.
   We do so at the board.
- One remedy for first problem: use many starting points and statistics to choose best local minimum.
- One remedy for second problem: use a better technique than GN or LM.
- Do Example 9.2 from the CD to illustrate some of these ideas.
- If time permits, do data fiting from Great Britian population data.

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- Recast: minimize  $\|\mathbf{Gm} \mathbf{d}\|_2$  unconstrained least squares.
- Recast: minimize  $\|G\mathbf{m} \mathbf{d}\|_2$  subject to  $\|L\mathbf{m}\|_2 \le \epsilon$ , where L is a damping matrix (e.g., L = I.)
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- Regularize:  $\|G(\mathbf{m}) \mathbf{d}\|_{2}^{2} + \alpha^{2} \|L\mathbf{m}\|_{2}^{2}$ .
- Equivalently: minimize  $\left\| \begin{bmatrix} G(\mathbf{m}) \mathbf{d} \\ \alpha L \mathbf{m} \end{bmatrix} \right\|_{2}^{2} \equiv \|H(\mathbf{m})\|_{2}^{2}$ .
- Linearize: Compute the Jacobian of this vector function:

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- Linearize  $||G(\mathbf{m}) \mathbf{d}||_2^2 + \alpha^2 ||L\mathbf{m}||_2^2$  by making the above replacement for  $G(\mathbf{m})$ . Call the solution  $\mathbf{m}^{k+1}$ .
- This leads to the system  $m^{k+1} = \left(\nabla G\left(\mathbf{m}^k\right)^T \nabla G\left(\mathbf{m}^k\right) + \alpha^2 L^T L\right)^{-1} \nabla G\left(\mathbf{m}^k\right)^T \widehat{\mathbf{d}}\left(\mathbf{m}^k\right),$  where  $\widehat{\mathbf{d}}\left(\mathbf{m}^k\right) = d G\left(\mathbf{m}^k\right) + \nabla G\left(\mathbf{m}^k\right)^T \mathbf{m}^k$ .
- The algorithm is to solve this equation with initial guess  $\mathbf{m}^0$ , but at each iteration choose the largest value of  $\alpha$  such that  $\chi^2\left(\mathbf{m}^{k+1}\right) \leq \delta^2$ . If none, pick value of  $\alpha$  that minimizes  $\chi^2$ . Stop if/when sequence converges to solution with  $\chi^2 \leq \delta^2$ .

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- This leads to the system  $m^{k+1} = \left(\nabla G\left(\mathbf{m}^k\right)^T \nabla G\left(\mathbf{m}^k\right) + \alpha^2 L^T L\right)^{-1} \nabla G\left(\mathbf{m}^k\right)^T \widehat{\mathbf{d}}\left(\mathbf{m}^k\right),$  where  $\widehat{\mathbf{d}}\left(\mathbf{m}^k\right) = d G\left(\mathbf{m}^k\right) + \nabla G\left(\mathbf{m}^k\right)^T \mathbf{m}^k$ .
- The algorithm is to solve this equation with initial guess  $\mathbf{m}^0$ , but at each iteration choose the largest value of  $\alpha$  such that  $\chi^2\left(\mathbf{m}^{k+1}\right) \leq \delta^2$ . If none, pick value of  $\alpha$  that minimizes  $\chi^2$ . Stop if/when sequence converges to solution with  $\chi^2 \leq \delta^2$ .

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