Exam 6 **Math 208** **Fall 2004**

Name: Key **Score:**________

Instructions: Show your work in the spaces provided below for full credit. Use the reverse side for additional space, *but clearly so indicate.* You must clearly identify answers and show supporting work to receive any credit. Exact answers (e.g., π) are preferred to inexact (e.g., 3.14). Point values of problems are given in parentheses. Notes or text in *any* form not allowed. Calculator is required.

(16) **1.** Let $\mathbf{F} = \langle 2x, 2yz^2, 2y^2z \rangle$.

(a) Show that $\mathbf{F}$ is conservative without actually finding a potential function for $\mathbf{F}$.

Since domain of $\underline{F}$ is simply connected., suffices to show $\nabla \times \underline{F} = \underline{0}$

8

$$\nabla \times \underline{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x & 2yx^2 & 2y^2x \end{vmatrix} = \langle 4yz - 4yz, -(0-0), 0-0 \rangle$$

$$= \underline{\underline{\langle 0,0,0 \rangle}}$$

So F is $\underline{\underline{\text{conservative}}}$

(b) Calculate $\nabla \cdot \mathbf{F}$.

8

$$\nabla \cdot \underline{F} = \frac{\partial}{\partial x} 2x + \frac{\partial}{\partial y} 2yz^2 + \frac{\partial}{\partial z} 2y^2 z$$

$$= 2 + 2z^2 + 2y^2$$

(12) **2.** Let $f(t)$ be a scalar function, $\mathbf{r} = \langle x, y \rangle$ and $r = \|\mathbf{r}\| = \sqrt{x^2+y^2}$. Show that $\nabla f(r) = f'(r)\frac{\mathbf{r}}{r}$.

4 $\nabla f(r) = \langle f(r)_x, f(r)_y \rangle$ Use Chain Rule: [or use identity $\nabla f(g) = f'$

4

$$= \langle f'(r)\, r_x, f'(r)\, r_y \rangle$$

$$= f'(r) \left\langle \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}, \frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}} \right\rangle$$

4

$$= f'(r) \frac{1}{\sqrt{x^2+y^2}} \langle x, y \rangle$$

$$= f'(r) \frac{\underline{r}}{r}$$

(24) **3.** Use the Divergence Theorem to evaluate $\int\int_S \mathbf{F}\cdot\mathbf{n}dS$, where $\mathbf{F} = \langle y^3 - 2x, e^{xz}, 4z\rangle$ and S is the boundary of the rectangular box $0 \le x \le 2$, $1 \le y \le 2$, $-1 \le z \le 2$, with exterior unit normal.

By Gauss Div. Thm $\displaystyle\iint_S \underline{F}\cdot\underline{n}\, dS = \iiint_Q \nabla\cdot\underline{F}\, dV$ 8

where Q is the rectangular box and $\underline{n}$ exterior normal.

Here $\nabla\cdot\underline{F} = \frac{\partial}{\partial x}(y^3 - 2x) + \frac{\partial}{\partial y} e^{xz} + \frac{\partial}{\partial z} 4z = -2 + 4 = +2$ 8

So $\displaystyle\iiint_Q \nabla\cdot\underline{F}\, dV = \iiint_Q 2\, dV = 2\iiint_Q dV$ 8

$= 2\cdot\text{volume of box}$

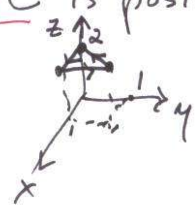
$= 2\cdot 2\cdot 1\cdot 3 = \underline{\underline{12}}$

(24) **4.** Use Stokes' Theorem to evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle \sin(x^2), y, z - y\rangle$ and curve C is the horizontal triangle from $(1, 0, 2)$ to $(1, 1, 2)$ to $(0, 0, 2)$ in that order.

Give 8pts if worked correctly.

By Stokes' Thm $\displaystyle\oint_C \underline{F}\cdot d\underline{r} = \iint_S \nabla\times\underline{F}\cdot\underline{n}\, dS$

where S is interior of the triangle and $\underline{n}$ chosen so that C is positively oriented. From picture we see that



$n = \underline{k} = \langle 0, 0, 1\rangle$. 4

Also $\nabla\times\underline{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin(x^2) & y & z-y \end{vmatrix} = \langle -1-0, -(0-0), 0-0\rangle$

$= \langle -1, 0, 0\rangle = -\underline{i}$.

So $\nabla\times\underline{F}\cdot\underline{n} = \underline{k}\cdot(-\underline{i}) = 0$. 4

Hence $\displaystyle\iint_S \nabla\times\underline{F}\cdot\underline{n}\, dS = 0$, so $\displaystyle\oint_C \underline{F}\cdot d\underline{r} = \underline{\underline{0}}$ 4

4

(24) **5.** Let a surface be given by $z = \sqrt{x^2+y^2}$ where $(x,y) \in R = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 2\}$.
(a) Find formulas for vector and scalar differential surface area $d\mathbf{S}$ and dS in terms of dA, differential surface area in the xy-plane.

Let $z = f(x,y) = \sqrt{x^2+y^2}$, and we get

$$d\underline{S} = \langle -f_x, -f_y, 1 \rangle dA$$
$$= \left\langle \frac{-\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}, \frac{-\cancel{2}y}{2\sqrt{x^2+y^2}}, 1 \right\rangle dA$$
$$= \underline{\left\langle -\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1 \right\rangle dA}$$

4

Thus $dS = \|d\underline{S}\| = \sqrt{\frac{(-x)^2}{x^2+y^2} + \frac{(-y)^2}{x^2+y^2} + 1}\, dA = \sqrt{\frac{x^2+y^2}{x^2+y^2}+1}\, dA = \underline{\sqrt{2}\, dA}$ 4

(b) Express $\iint_S f(x,y,z)dS$ as an iterated integral in x and y where $f(x,y,z) = \sin(xyz^2)$. Do *not* work the integral out.

Here [redacted] -2 if z not replac

$$\iint_S \sin(xyz^2)\, dS = \iint_R \sin(xy(x^2+y^2))\sqrt{2}\, dA$$
$$= \int_0^2 \int_0^1 \sin(xy(x^2+y^2))\sqrt{2}\, dx\, dy$$

(c) Express $\iint_S \mathbf{F} \cdot d\mathbf{S}$ as a double integral over R, where $\mathbf{F} = \langle x, 0, y^2 z \rangle$. Do *not* work the integral out.

Here $\underline{F} \cdot d\underline{S} = \langle x, 0, y^2 z \rangle \cdot \left\langle -\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1 \right\rangle dA$

$$= \left(\frac{-x^2}{\sqrt{x^2+y^2}} + y^2 z\right) dA$$
$$= \left(y^2\sqrt{x^2+y^2} - \frac{x^2}{\sqrt{x^2+y^2}}\right) dA$$ -2 if z not replaced

So $\iint_S \underline{F} \cdot d\underline{S} = \underline{\iint_R \left(y^2\sqrt{x^2+y^2} - \frac{x^2}{\sqrt{x^2+y^2}}\right) dA}$

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