Exam 3 **Math 208** **Fall 2004**

Name: Key **Score:** 100/100

Instructions: Show your work in the spaces provided below for full credit. Use the reverse side for additional space, *but clearly so indicate.* You must clearly identify answers and show supporting work to receive any credit. Exact answers (e.g., π) are preferred to inexact (e.g., 3.14). Point values of problems are given in parentheses. Notes or text in *any* form not allowed. Calculator is required.

(24) **1.** Let $f(x, y, z) = 3x^2y - z\cos x$.

(a) Find $\nabla f(0, 2, -1)$.

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$$\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle = \langle 6xy + z\sin(x), 3x^2, -\cos x \rangle$$

$$\text{So } \nabla f(0,2,-1) = \langle 0 + (-1)\cdot 0, 0, -1 \rangle = \underline{\langle 0, 0, -1 \rangle}$$

(b) Find a unit vector in the direction of the maximum directional derivative of f at the point $(0, 2, -1)$.

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The direction of maximum rate of change is that of ∇f. So the answer is

$$\frac{\nabla f(0,2,-1)}{\|\nabla f(0,2,-1)\|} = \frac{\langle 0,0,-1\rangle}{1} = \underline{\langle 0,0,-1\rangle}$$

(c) Find an equation for the tangent plane to the surface $f(x, y, z) = 1$ at the point $(0, 2, -1)$.

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From part (a) the answer is

$$0\cdot(x-0) + 0\cdot(y-2) - 1(z - -1) = 0$$

That is, $\underline{z = -1}$

(18) **2.** Given a function $g(u, v) = f(x(u, v), y(u, v))$ with all continuous partials, find $\partial^2 g/\partial u^2$ in terms of f, x, y and their partials.

7 $\quad \dfrac{\partial g}{\partial u} = \dfrac{\partial}{\partial u} f(x,y) = f_x x_u + f_y y_u$

5 $\quad \text{So } \dfrac{\partial^2 g}{\partial u^2} = \dfrac{\partial}{\partial u}(f_x x_u) + \dfrac{\partial}{\partial u}(f_y y_u)$

(don't have to simplify)

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$$= (f_{xx} x_u + f_{xy} y_u) x_u + f_x x_{uu} + (f_{yx} x_u + f_{yy} y_u) y_u +$$

$$= \underline{f_{xx} x_u^2 + 2f_{xy} x_u y_u + f_x x_{uu} + f_{yy} y_u^2 + f_y y_{uu}}$$

(33) **3.** Let $f(x, y) = 3x^3 + y^2 - 9x + 4y$.

(a) Find all critical points of f. f is smooth, so

set $0 = f_x = 9x^2 - 9$
$0 = f_y = 2y + 4$ } 6

Get $y = -2$, and $x^2 = 1$, so $x = \pm 1$
Critical points: $(1,-2), (-1,-2)$. } 6

(b) Use the second derivative test to classify the critical points of f.

$f_{xy} = 0$. $f_{xx} = 18x$, $f_{yy} = 2$.
So $Df(a,b) = f_{xx} f_{yy} - f_{xy}^2 = 36x$. } 4

4 { Now $Df(1,-2) = 36 \cdot 1 > 0$. So have local max/min here. But $f_{yy}(1,-2) = 2 > 0$, so it is a local min.

4 { $Df(-1,-2) = -36 < 0$. So have a saddle pt. here

(c) Suppose the domain of $f(x, y)$ is restricted to the square $0 \le x, y \le 2$. Does $f(x, y)$ have a global maximum or minimum on this domain? In either case, explain your answer and briefly indicate how you would find them if an answer is affirmative.

5 { The fcn is cnt. on closed bounded set, hence must assume its global max/mins in the set, namely at interior critical points or on bdy.

5, 4 { So we would find the critical points interior to the square, then search along the 4 line boundaries for extrema using 1 variable methods. Finally, we would evaluate f at all these candidate points and pick extrema.

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(25) **4.** Let $f(x,y) = 4xy$.

(a) Find the extrema of f subject to the constraint $4x^2 + y^2 = 8$ by the method of Lagrange multipliers.

Let $g(x,y) = 4x^2 + y^2 - 8$ so constraint is $g(x,y) = 0$.

L.M. Equations: $\langle f_x, f_y \rangle = \lambda \langle g_x, g_y \rangle$, $g = 0$,

i.e.,

$$4y = 8\lambda x.$$
$$4x = 2y\lambda$$
$$4x^2 + y^2 = 8.$$

If $x = 0$, Then $4y = 0$, so $y = 0$ which is impossible since $4x^2 + y^2 = 8$. And $y=0$ gives $x=0$.

So $x, y \neq 0$.

Eliminate λ: $\frac{4y}{8x} = \lambda = \frac{4x}{2y}$, so $\frac{y}{2x} = \frac{2x}{y}$, ie $y^2 = 4x^2$.

Thus $4x^2 + y^2 = 2y^2 = 8$, $y^2 = 4$, $y = \pm 2$.

For each y, $x^2 = \frac{y^2}{4} = 1$, so $x = \pm 1$.

Get 4 critical points. Evaluate to find extreme values:

$$f(1,2) = 4 \cdot 2 = 8$$
$$f(-1,2) = -4 \cdot 2 = -8$$
$$f(1,-2) = -4 \cdot 2 = -8$$
$$f(-1,-2) = 4 \cdot 2 = 8$$

So $\underline{8}$ is maximum value and $\underline{-8}$ is minimum values of f, sbj to $g = 0$.

(b) Find the absolute extrema of f over the region $4x^2 + y^2 \leq 8$. Use EVT:

In addition to above, have to find all critical interior pts of $f(x,y)$ and test Them.

$$\left.\begin{aligned} f_x &= 4y = 0 \\ f_y &= 4x = 0 \end{aligned}\right\} \text{ gives } x = y = 0.$$

But $f(0,0) = 0$ which is neither max nor min,

So max & min values over whole region $4x^2 + y^2 \leq 8$ are same as in (a):

Max value of f: 8

Min value of f: $\underline{-8}$

-3 if 2nd deriv used w constraints.

-2 if some discussion thereafter

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