**Exam 1** Math 208 Fall 2004

Name: Key Score: 100/100

*Instructions:* Show your work in the spaces provided below for full credit. Use the reverse side for additional space, *but clearly so indicate.* You must clearly identify answers and show supporting work to receive any credit. Exact answers (e.g., $\pi$) are preferred to inexact (e.g., 3.14). Point values of problems are given in parentheses. Notes or text in *any* form not allowed. Calculator is required.

(32) **1.** Given the vectors $\mathbf{a} = \mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = -2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{c} = \mathbf{j} + \mathbf{k}$, find the following:

(a) A unit vector in the direction of $\mathbf{a}$.

$$\underline{u} = \frac{\underline{a}}{\|\underline{a}\|} = \frac{1}{\sqrt{1^2+4^2+5^2}}\langle 1,4,5\rangle = \underline{\underline{\frac{1}{\sqrt{42}}\langle 1,4,5\rangle}} \left(\frac{1}{\sqrt{42}}(\underline{i}+4\underline{j}+5\underline{k})\right)$$

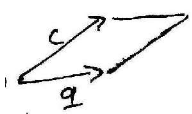
(b) The angle between $\mathbf{a}$ and $\mathbf{b}$.

$$\cos\theta = \frac{\underline{a}\cdot\underline{b}}{\|\underline{a}\|\cdot\|\underline{b}\|} = \frac{1(-2)+4\cdot 1+5\cdot 2}{\sqrt{42}\sqrt{(-2)^2+1^2+2^2}} = \frac{12}{3\sqrt{42}} = \frac{4}{\sqrt{42}}$$

2 So $\theta = \cos^{-1}\left(\frac{4}{\sqrt{42}}\right) \doteq \underline{\underline{51.88^\circ}}$ ($\doteq 0.9056$ rad)

(c) The area of a parallelogram with adjacent sides formed by representatives of $\mathbf{a}$ and $\mathbf{c}$.

This area is $\|\underline{a}\times\underline{c}\| = \left\|\begin{vmatrix} i & j & k \\ 1 & 4 & 5 \\ 0 & 1 & 1\end{vmatrix}\right\| = \|\langle 4-5, -(1-0), 1\rangle\|$




$$= \sqrt{(-1)^2+(-1)^2+1^2} = \underline{\underline{\sqrt{3}}}$$

(d) $\text{proj}_{\mathbf{b}}\mathbf{a}$ and $\text{comp}_{\mathbf{b}}\mathbf{a}$.

1 $$\text{proj}_{\underline{b}}\underline{a} = \frac{\underline{a}\cdot\underline{b}}{\underline{b}\cdot\underline{b}}\underline{b} = \frac{12}{9}\langle -2,1,2\rangle = \underline{\underline{\frac{4}{3}\langle -2,1,2\rangle}}$$

4 $$\text{comp}_{\underline{b}}\underline{a} = \frac{\underline{a}\cdot\underline{b}}{\|\underline{b}\|} = \frac{12}{3} = \underline{\underline{4}}$$

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(36) **2.** You are given two straight line paths

$$P_1: \begin{cases} x = 3 \\ y = 6-2t \\ z = 3t+1 \end{cases} \quad \text{and } P_2: \begin{cases} x = 1+2s \\ y = 3+s \\ z = 2+2s \end{cases}$$

(a) Show that these lines intersect.

We solve system $3 = 1+2s$   Get $s=1$ from $1^{st}$ eq'n.
$6-2t = 3+s$   So $2^{nd}$ & $3^{rd}$ are $6-2t=4$, $t=1$
$3t+1 = 2+2s$.   $3t+1=4$ $t=1$.

9 So these are consistent with $t=1$ and $s=1$.

Point of intersection is $\underline{\underline{(3,4,4)}}$.

(b) Find a vector orthogonal to both lines.

$\underline{a} = \langle 0,-2,3 \rangle$ is // to $P_1$ (2)
$\underline{b} = \langle 2,1,2 \rangle$ is // to $P_2$. (2)

So $\underline{a} \times \underline{b}$ desired vector.

9 $\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ 0 & -2 & 3 \\ 2 & 1 & 2 \end{vmatrix} = \langle (-2\cdot2-3), -(0-6), (0--4) \rangle = \langle -7, 6, 4 \rangle$ (5)

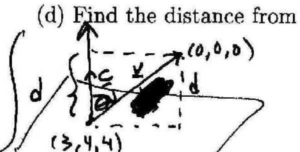
(c) Find the equation of a plane containing both lines.

6 Such a plane has $\underline{a} \times \underline{b}$ as a normal vector and contains pt $(3,4,4)$ of (a).

So eq'n is $-7x+6y+4z+d=0$

3 with $-7\cdot3+6\cdot4+4\cdot4+d=0$, $d=-19$.   So line is $\underline{\underline{-7x+6y+4z-19=0}}$

(d) Find the distance from this plane to the origin. Point on plane is $(3,4,4)$.



$\underline{c} = \underline{a} \times \underline{b} = \langle -7,6,4 \rangle$
$\underline{v} = \langle 0-3, 0-4, 0-4 \rangle$
$= \langle -3,-4,-4 \rangle$

9 $\dfrac{d}{\|\underline{v}\|} = \cos\theta$.   So $d = \left| \|\underline{v}\| \cos\theta \right| = \left| \dfrac{\underline{c}\cdot\underline{v}}{\|\underline{c}\|} \right| = \dfrac{-7(-3)+6(-4)+4(-4)}{\sqrt{(-7)^2+6^2+4^2}}$

[OR simply: $d = \left|\text{comp}_{\underline{c}} \underline{v}\right| = \left| \dfrac{\underline{c}\cdot\underline{v}}{\|\underline{c}\|} \right|$]   $= \dfrac{19}{\sqrt{101}} \doteq 1.89$

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(24) **3.** Let the position vector for a curve be given by $\mathbf{r}(t) = \langle 1, t, t^2 \rangle$. Calculate the following.

(a) $\mathbf{r}'(t)$ and $\int \mathbf{r}(t)\,dt$.

$\underline{r}(t) = \langle 1, t, t^2 \rangle$

4 $\{$ $r'(t) = \langle 1', t', t^{2\prime} \rangle = \underline{\langle 0, 1, 2t \rangle}$

4 $\{$ $\int \underline{r}(t)\,dt = \underline{\langle t, \frac{t^2}{2}, \frac{t^3}{3} \rangle + \underline{C}}$ $\left[\text{or } \langle t + c_1, \frac{t^2}{2} + c_2, \frac{t^3}{3} + c_3 \rangle\right]$

(2 pts for correct $\underline{C}$).

(b) Values of $t$ at which $\mathbf{r}(t) \perp \mathbf{r}'(t)$.

8 $\{$ Set $0 = \underline{r}(t) \cdot \underline{r}'(t) = \langle 1, t, t^2 \rangle \cdot \langle 0, 1, 2t \rangle = t + 2t^3$.

Get $0 = t(1 + 2t^2)$, so $t = 0$ is only soln.

Hence only such value is $\underline{t = 0}$

(c) Arc length of the portion of the curve from $t = 0$ to $t = 2$ (you may use your calculator).

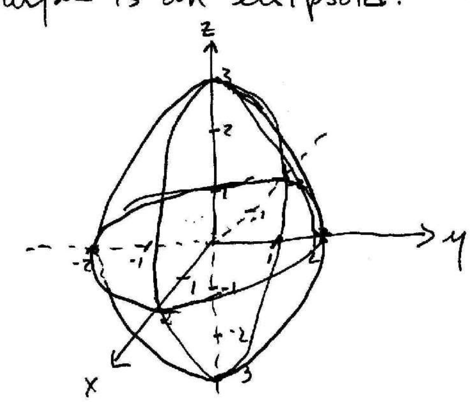
4 $\{$ arc length $= \int_{t=0}^{2} ds = \int_0^2 \sqrt{x'^2 + y'^2 + z'^2}\,dt = \int_0^2 \sqrt{0^2 + 1^2 + (2t)^2}\,dt = \int_0^2 \sqrt{1 + 4t^2}\,dt$

4 $\{$ $\doteq 4.64678$

OR: Exact antideriv $= \left\{\frac{1}{2} t \sqrt{1 + 4t^2} + \frac{1}{4} \operatorname{arcsinh}(2t)\right\}\Big|_0^2 = \sqrt{17} + \frac{1}{4} \operatorname{arcsinh}(4)$.

(8) **4.** Identify this surface and roughly sketch its $\underbrace{\text{traces}}_{4}$ and $\underbrace{\text{graph}}_{2}$: $\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{9} = 1$.

2 $\{$ Surface is an ellipsoid.

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