

## Math 208 Differentials Handout

The definition of the total differential of  $f$  on page 752 is not the general definition, but an application of the general definition to the task of approximating changes in a function. The total differential of a function is frequently referred to without having a specific point  $(x_0, y_0)$  to apply it to.

The total differential is very close to the chain rule in structure. For a function of two or more independent variables, the total differential of the function is the sum over all of the independent variables of the partial derivative of the function with respect to a variable times the total differential of that variable. The precise formula for any case depends on how many and what the variables are. Thus we get the following examples of formulas:

Given a function  $f(x, y)$ , its total differential is  $df = f_x dx + f_y dy$ .

Given a function  $f(x, s, w)$ , its total differential is  $df = f_x dx + f_s ds + f_w dw$ .

Given a function  $g(r, s, x, y, \lambda)$ , its total differential is

$$dg = g_r dr + g_s ds + g_x dx + g_y dy + g_\lambda d\lambda.$$

The structural relationship of the total differential to the chain rule is most obvious if we switch to  $\partial$  notation. For example, the total differential of  $f(x, y, z)$  in  $\partial$  notation is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

But if we think of  $x, y$ , and  $z$  as functions of  $t$ , the chain rule says

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}.$$

Note  $df$  looks like what you get if you multiply this last formula through by  $dt$  and cancel. As a specific example, if  $f(x, y, z) = xy^3 + z \ln(x) - 7$ , we get

$df = (y^3 + \frac{z}{x})dx + 3xy^2 dy + \ln(x) dz$ . (Note the absence of subscripts on  $x, y$ , and  $z$ .) The book in fact uses this more general notion of differential without explanation in problem 48 (which is assigned). When the differential is used to approximate changes in a function, the definition on page 752 can be used. Thus given  $f(x, y, z)$ , when evaluating  $df$  for the specific purpose of approximating the change in  $f$ , we let  $x = x_0$ ,  $y = y_0$ ,  $z = z_0$ ,  $dx = \Delta x$ ,  $dy = \Delta y$ , and  $dz = \Delta z$ , where  $(x_0, y_0, z_0)$  is the starting point and the final point is  $(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z)$ . However, we will see in Chapter 14 that when the more general version of  $df$  comes up inside a line integral, very nice things happen. So the total differential is significant beyond the application to approximating changes.

**Math 208**  
**Exercises on total differentials**

In exercises 1– 4 below, find the total differential of the given function.

1.  $f(x, w) = we^{x/w}$

2.  $f(x, y, z) = x^3 - ye^{4z} + 5$

3.  $g(y, z) = y^3 - 3\frac{z^2}{y}$

4.  $h(x, z, s, t) = sx^{-1} + 2zt^{-1} + 3xs^{-1} + 4tz^{-1}$ .

5. A right triangle is located on terrain where you can measure the hypotenuse and one leg directly, and you are also interested in the length of the other leg. Your measurement of the hypotenuse is 26 m. and of the leg is 24 m.
- a) What is your estimate for the length of the other leg?
  - b) If your hypotenuse measurement could be off by up to 0.4 m, and your leg measurement could be off by up to 0.2 m, use the total differential to estimate what's the most you're likely to be off on the length of the calculated leg.

Answers to the exercises on the Differentials Handout:

$$1. \quad df = e^{x/w} dx + \left( e^{x/w} - \frac{x}{w} e^{x/w} \right) dw$$

$$2. \quad df = 3x^2 dx - e^{4z} dy - 4ye^{4z} dz$$

$$3. \quad dg = (3y^2 + 3z^2 y^{-2}) dy - \frac{6z}{y} dz$$

$$4. \quad dh = (-sx^{-2} + 3s^{-1}) dx + (2t^{-1} - 4tz^{-2}) dz + (x^{-1} - 3xs^{-2}) ds + (4z^{-1} - 2zt^{-2}) dt$$

5. a) 10 m.  
b) 1.52 m.