JDEP 384H: Numerical Methods in Business

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Lecture 10, February 8, 2007
110 Kaufmann Center

Outline

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Immunization Strategies

Example

Use Matlab to determine the correct weighting of three bonds with durations $2, 4, 6$ and convexities $12, 15, 20$, respectively, if we are to shape a portfolio with duration 3 and convexity 13.

Solution. Use the First Pass strategy. Work this system out with Matlab. What happens if no short positions are allowed?
What about the Second Pass?
With 3 bonds, we're stuck. But increase the number by, say one, to 4 bonds. Now we have a new situation of 3 equations in 4 unknowns. Since unknowns exceed equations, we can expect infinitely many solutions if any at all (see LinearAlgebraLecture)!
So which do we select?

Idea: Use the extra degree(s) of freedom to turn the problem into a linear programming problem. For example, maximize the weighted yield of the portfolio. Say the bonds have yields Y_1, Y_2, Y_3, Y_4. The problem becomes: Maximize the objective function

$$f(w_1, w_2, w_3, w_4) = Y_1 w_1 + Y_2 w_2 + Y_3 w_3 + Y_3 w_3 + Y_4 w_4$$

subject the the three constraints as in first pass with four variables.

An example portfolio consists of:

a weighted combination (no short positions) of four bonds with durations 2, 3, 4, 6, convexities $12, 12.5, 15, 20$, and yields 0.06, 0.061, 0.065, 0.07, respectively. How to maximize yield?

- Use Matlab to solve this problem. If there is a solution, what is the maximum yield?
- What if we relax the convexity constraint to having convexity at least 13? Portfolio yield?
- What if we drop the convexity constraint altogether? Portfolio yield and convexity?

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Quantifying Risk and Risk Tolerance

We examine portfolios of **risky securities**, such as stocks.
Note: In fact, bonds have an element of risk too!

Risk:

How can we measure risk?

- The return on our investment is wealth X, which is now a random variable. So are the returns R_i of each stock in our portfolio.
- As such, returns have an **expected value** (mean) $x = E[X]$ which is the weighted sum of the expected returns r_i of the ith stock.
- Variability of a r.v. is measured by its standard deviation. Hence the **risk** of the ith stock is just $\sigma_i = \sqrt{\mathrm{Var}(R_i)}$.
- Competing goals: maximize return, minimize risk.

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So What Do You Want?

One way to quantify an investor's preferences;

Utility Function $u(x)$ of payoff x:

A numerical measure of satisfaction gained from a payoff x.

- Normally, u is monotone increasing with x.
- Normally, u is concave (concave down) implies that $u''(x) < 0$, which implies that $u'(x)$ is decreasing.
- Hence concavity is a measure of risk aversion, because it implies that each increment to wealth conveys progressively smaller increments to utility.
- Two examples: $u(x) = \log x$ and $u(x) = x - \frac{\lambda}{2}x^2$ $(x \leq 1/\lambda)$.

There are other types of utility functions, e.g., we could have u depend on the expected rate of return and variance

$$u = r - 0005 \cdot A \cdot \sigma^2$$

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Consequences of Utility Approach:

- Obtain measures of degree of risk aversion (Arrow-Pratt absolute and relative risk aversion coefficients):

$$R_u^a(x) = -\frac{u''(x)}{u'(x)} \text{ and } R_u^r(x) = -\frac{u''(x)\,x}{u'(x)}$$

- Portfolio optimization becomes a math problem: Given initial wealth W_0, a set of assets with return R_i (a random variable), and portfolio with x_i dollars of ith asset,

$$\max E\left[u\left(x_1 R_1 + \cdots + x_n R_n\right)\right]$$

subject to $x_1 + \cdots + x_n = W_0$.

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Mean-Variance Portfolio Optimization

Following text, we'll stick to a portfolio of two risky assets for purpose of illustration. Rather than use absolute quantities, we use rates of return. For simplicity, examine a portfolio of two assets.

- The absolute x_i above are replaced by fractions w_i, where $w_1 + w_2 = 1$ and we denote the vector $[w_1; w_2]$ by $\mathbf{w}$.
- Assume no short positions, so the $w_i \geq 0$.
- Rates of return on investments are r_1, r_2, respectively, so rate of return of portfolio is $r = w_1 r_1 + w_2 r_2$.
- The expected returns are $\bar{r}_1$, $\bar{r}_2$ and $\bar{r} = w_1 \bar{r}_1 + w_2 \bar{r}_2$.
- The r.v.'s r_1, r_2 have covariance matrix Σ, so that the variance of our portfolio is

$$\mathrm{Var}\left(w_1 r_1 + w_2 r_2\right) = \mathbf{w}^T \Sigma \mathbf{w}.$$

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Mean-Variance Portfolio Optimization

Problems:

- For a given expected return $\bar{r}_T$, what weighting gives the minimum variance?
- Answer: the solution to the quadratic programming problem:

$$\min \mathbf{w}^T \Sigma \mathbf{w}$$

subject to

$$\begin{aligned} \mathbf{w}^T \bar{\mathbf{r}} &= \bar{r}_T \\ \mathbf{w}^T \mathbf{1} &= 1 \end{aligned}$$

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Definition

A portfolio is **efficient** if it is not possible to obtain a higher expected return without increasing the risk.

Definition

An **efficient frontier** is a graph of efficient portfolio's risk versus expected return.

Example

From text, p. 74, suppose two assets have expected earning rates $\bar{r}_1 = 0.2$, $\bar{r}_2 = 0.1$, $\sigma_1^2 = 0.2$, $\sigma_2^2 = 0.4$ and $\sigma_{12} = -0.1$. Design an efficient frontier for this problem using Matlab. How would we find the leftmost point on the graph?

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