Final Exam Key, JDEP 384H, Spring 2006

Due Date for Exam: Thursday, May 4, 12:00 noon.

Instructions: Show your work and give reasons for your answers. Write out your solutions neatly and completely. There is to be absolutely no consultation of any kind with anyone else other than me about the exam. If there are points of clarification or corrections, I will post them on our message board. ALL materials used in your work that have not been provided by me for this course must be explicitly credited in your write-up. Point values are indicated. You may send an email document (preferably a pdf file, but Word documents will be accepted) or hand in hard copy at my office. Be sure identify yourself as the owner of any document you turn in to me, electronic or hard copy. Point values of problems are indicated for a total of 130 points.

(18 pts) **1.** Define eigenvalue and give two instances where the eigenvalues of a matrix contain useful information.

SOLUTION. (8 pts) An eigenvalue of the square $n \times n$ matrix A is a number λ for which there exists a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda \mathbf{x}$.

(5 pts each): describe an application such as convergence of iterative process, convergence of numerical PDE method, testing positive definiteness of symmetric matrix.

(18 pts) **2.** Explain what the differences between American and European call option boundary conditions are and why they differ.

SOLUTION. (6 pts, background) A European call option is a contract giving the owner the right to buy a number of shares of a stock from the option writer at a fixed price K (per share), called the strike price, at a fixed time $t = T$ in the future, called the expiry date. Here time $t = 0$ is the time of sale of the option. The only difference between an American and European option is that the owner of an American option may exercise it at any time up to expiry. The stock in question has price S (per share) as a function of time and may or may not pay dividends, which are assumed to be continuous at a rate D_0. Let $C(S,t)$ be the price of the option at time t with the stock price at S.

(6 pts, bc's) We choose a large right boundary value S_{max} which is the practical upper bound on the stock price before expiry, since theoretically there is no bound on stock price. Boundary conditions for a European option are

- Left boundary condition: $C(0,t) = 0$.
- Right boundary condition:

$$C(S_{max},t) = S_{max}e^{-D_0(T-t)} - Ke^{-r(T-t)}.$$

Boundary conditions for an American option are

- Left boundary condition: $C(0,t) = 0$.

Right boundary condition:

$$P(S_{max},t) = \max\left\{S_{max} - K, S_{max}e^{-D_0(T-t)} - Ke^{-r(T-t)}\right\}.$$

(6 pts, difference) Left boundary conditions agree because if the value of the stock is zero, a right to purchase at any time is worthless. The reason for the right boundary condition for the European option is that the value of the stock has to be discounted from expiry to present time t due to payment of dividend rates and the strike price K is discounted over the same time period to give the present value of the option

price. The reason for the difference in American option is that firstly, the American option should always have value at least as great as the price of a European option since it may be exercised at earlier times and secondly, this value should always at least match the payoff curve since it will be executed if the price hits the payoff curve to ensure no loss. There is no discounting from the payoff curve since the American option will be exercised at an earlier date if the payoff curve is met or for that matter, any other reason.

(18 pts) **3.** Exhibit and explain the meaning of the error term in Crank-Nicolson method, given that it converges. In particular, if you computed with a particular step size for x and t and wanted to halve the error, how would you change the step size?

SOLUTION. (10 pts) The error term is given in the notes for Crank-Nicolson (given that it converges) is given to be

$$u_{true}\left(x_i, t_j\right) - u_{i,j} = \mathcal{O}\left(\delta t^2 + \delta x^2\right).$$

Here $u_{i,j}$ is our approximation to $u_{true}\left(x_i, t_j\right)$ computed by the Crank-Nicolson method. What this means is that there is a constant K, independent of step sizes δt, δx in time and space and indices i, j such that

$$\left|u_{true}\left(x_i, t_j\right) - u_{i,j}\right| \leq K\left(\delta t^2 + \delta x^2\right)$$

as $\delta t, \delta x \to 0$.

(8 pts) In particular, this means that if we take the inequality to be an approximate equality, and we want to cut the error in half, then all we have to do is reduce both δt and δx by a factor of $1/\sqrt{2}$, since

$$K\left(\left(\frac{\delta t}{\sqrt{2}}\right)^2 + \left(\frac{\delta x}{\sqrt{2}}\right)^2\right) = \frac{1}{2}K\left(\delta t^2 + \delta x^2\right),$$

which cuts the error in half.

(16 pts) **4.** We use the terms "forward" and "backward" time in discussing Black-Scholes equations for option prices. Explain what this means and how it is a useful idea.

SOLUTION. (6 pts) The term "forward" time refers to real time, call it τ, where we measure time from the purchase time of the option, $\tau = 0$ forward to expiry time $\tau = T$. On the other hand "backward" time t is given by the formula $t = T - \tau$.

(10 pts) The reason for introducing backwards time is that the option price $f(S, \tau)$ does not have the correct PDE context for solving this problem. Specifically, the value of the option is unknown at real time $\tau = 0$, so it is not possible to have initial conditions, which are needed in order to solve the Black-Scholes PDE as an IBVP. However, backwards time $t = 0$ corresponds to real time $\tau = T$, expiry time. The value of the option is known at that time to be the value of the payoff curve. Hence changing variables from (S, τ) to (S, t) allows us an initial condition for the PDE. Thus the problem becomes a well-posed problem. It also changes Black-Scholes to a PDE that is very much like the heat equation.

(20 pts) **5.** Consider a European call option on a dividend paying stock with volatility $\sigma = 0.4$, dividend rate $D_0 = 0.04$, risk-free interest rate $r = 0.06$ and time to expiry $T = 6/12$, six months. Plot the price of the option at six, four and two months before expiry. Locate (approximately) where the stock crosses the payoff curve at those three times. As you move close to expiry, how does the crossover point appear to move?

SOLUTION. The strike price will be assumed to be 50, since it was not specified. We ran this script to get a plot for zooming, which does give ok answers, but found that it was more accurate to use Matlab's find function to determine the crossover point. You could have done the same thing by simply inspecting the vector `C - Payoff` below and looking for the change in sign. Script and graph follow:

```
% script:  finexer_5.m
disp('Exercise 5:  European call with dividends')
K = 50
```

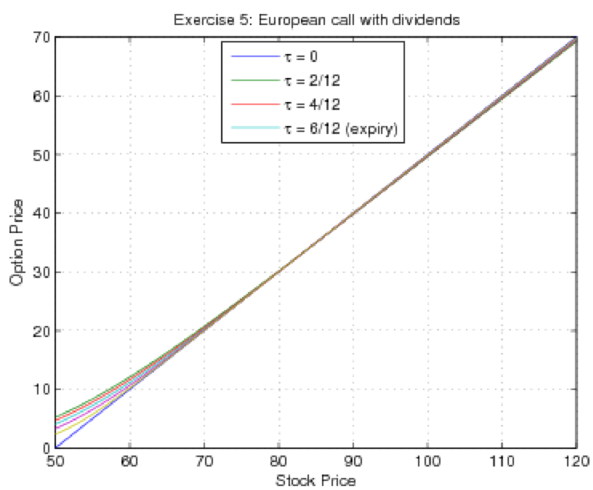
```
sigma = 0.4
r = 0.06
D0 = 0.04
T = 6/12
M = 721
Smax = 120
Snodes = linspace(50,Smax,M)'; % can't cross payoff curve below S=50
Payoff = max(Snodes-K,0);
figure(1)
plot(Snodes,Payoff);
grid, hold all
Cross = zeros(5,3);
for j = 1:5
C = bseurcall(Snodes,K,r,T,j/12,sigma,D0);
ndx = find(C<Payoff);
disp('Time:')
disp((j-1)/12)
Cross(j,1) = (j-1)/12;
disp('Crossover node:')
disp(Snodes(ndx(1)))
Cross(j,2) = Snodes(ndx(1));
disp('Option value:')
disp(C(ndx(1)))
Cross(j,3) = C(ndx(1));
plot(Snodes,C);
end
title('Exercise 5:  European call with dividends');
xlabel('Stock Price')
ylabel('Option Price')
legend('\tau = 0','\tau = 2/12', '\tau = 4/12', '\tau = 6/12 (expiry)');
figure(2)
grid,hold all
plot(Cross(:,1),Cross(:,2:3))
title('Exercise 5:  Crossover Points and Option Price')
xlabel('Time')
ylabel('Crossover Point/Option')
```

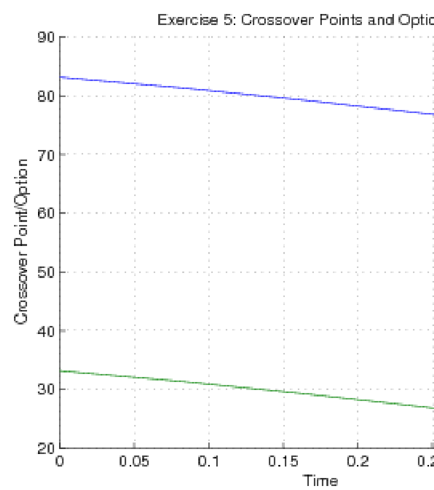
The output from the script is as follows:

```
finalexer_5
Exercise 5:  European call with dividends
K =
50
sigma =
0.4000
r =
0.0600
D0 =
0.0400
T =
0.5000
M =
721
Smax =
```

```
120
Time:
0
Crossover node:
83.1528
Option value:
33.1496
Time:
0.0833
Crossover node:
81.3056
Option value:
31.3035
Time:
0.1667
Crossover node:
79.1667
Option value:
29.1657
Time:
0.2500
Crossover node:
76.8333
Option value:
26.8328
Time:
0.3333
Crossover node:
75.0833
Option value:
25.0832
```

Graphs generated:





We conclude from this output that that the crossover point and the price of the option are both decreasing as we move towards expiry. One could graph both against time and see that the movement of crossover point and price at crossover are not quite linear, but nearly so.

(20 pts) **6.** Use `CrankNicolsonSORLC.m` to find the price of an American put five months before expiry, where the strike price is $K = 50$, volatility $\sigma = 0.35$, risk-free interest rate is 5.5%. Do the same with `ExplicitEulerLC.m` and find suitable parameters that give a reasonably close plot. Then plot the difference between the two answers.

SOLUTION. Here is the script used to generate the the graphs that follow. Since different values for dt were used, the output of the script is also listed.

The script:

```
% script:  finalexer_6.m
disp('Exercise 6, Final:  American put')
global K;
global sigma;
global r;
```

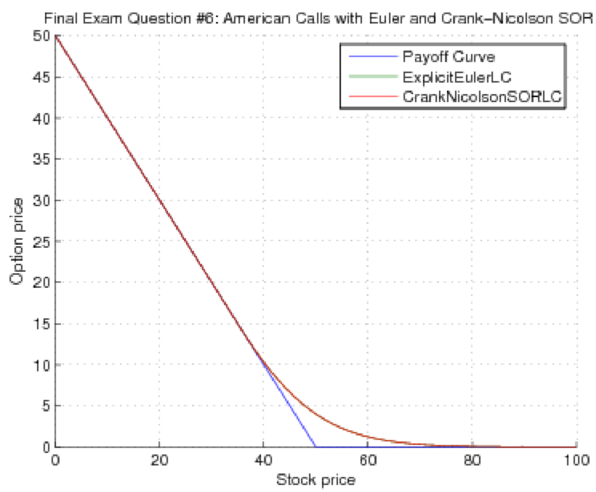
```
global xmax;
global xmin;
global D0
K = 50
sigma = 0.35
r = 0.055
xmax = 100
xmin = 0
D0 = 0.0
M = 200
dx = (xmax-xmin)/M
xnodes = linspace(xmin,xmax,M+1)';
T = 5/12
dt = dx^2/2400 % for explicit Euler
NE = round(T/dt)
% now adjust dt so that T=N*dt more accurately
dt = T/NE
% now fix needed functions using globals before proceeding
solnE = ExplicitEulerLC(xmin,xmax,M,NE,dt,'cDiffus','vVelocity','kRate','fLeftBdy', 'gRightBdy',
dt = T/M; % for Crank Nicolson SOR
NC = round(T/dt)
% now adjust dt so that T=N*dt more accurately
dt = T/NC
tol = 0.0001
omega = 1.4
solnC = CrankNicolsonSORLC(xmin,xmax,M,NC,dt,'cDiffus','vVelocity','kRate','fLeftBdy', 'gRightBd
figure(1),grid,hold on
plot(xnodes,[solnE(:,[1,NE+1]),solnC(:,NC+1)])
title('Final Exam Question #6:  American Calls with Euler and Crank-Nicolson SOR')
xlabel('Stock price')
ylabel('Option price')
legend('Payoff Curve','ExplicitEulerLC','CrankNicolsonSORLC');
figure(2),grid,hold off
plot(xnodes,solnE(:,NE+1) - solnC(:,NC+1))
title('Difference Between Euler and Crank-Nicolson SOR Values')
xlabel('Stock price at time 0')
ylabel('Difference')
grid
```

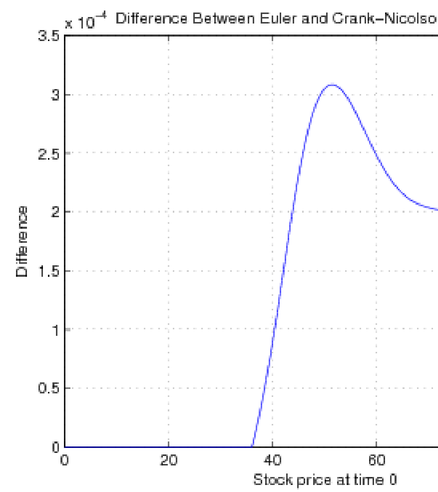
The output:

```
finalexer_6
Exercise 6, Final:  American put
K =
50
sigma =
0.3500
r =
0.0550
xmax =
100
xmin =
0
D0 =
```

```
0
M =
200
dx =
0.5000
T =
0.4167
dt =
1.0417e-004
NE =
4000
dt =
1.0417e-004
NC =
200
dt =
0.0021
tol =
1.0000e-004
omega =
1.4000
```

The graphs:





(20 pts) **7.** Use sample mean Monte Carlo integration with unmodified variates to estimate

$$\int_{-\infty}^{\infty} \frac{\sin^2(x)}{1+x^2} e^{-x^2/2}\, dx$$

to an error at most 0.01 at the 95% confidence level. Calculate the confidence interval.

SOLUTION. We follow the instructions outlined in the notes, which are as follows:

- Write integral as $\int_a^b g(x)\, dx = \int_a^b \frac{g(x)}{f(x)} f(x)\, dx$ where $f(x)$ is known positive p.d.f. which vanishes outside $[a,b]$.
- Interpret $\int_a^b g(x)\, dx = E\left[\frac{g(X)}{f(X)}\right]$ where X is r.v. with p.d.f. $f(x)$.
- Take N independent samples of X and deduce $\int_a^b g(x)\, dx \approx \frac{1}{N}\sum_{i=1}^{N} \frac{g(X_i)}{f(X_i)}$.

In our situation, the standard normal distribution $N(0,1)$ with p.d.f. $f(x) = e^{-x^2/2}/\sqrt{2\pi}$ on the interval $(-\infty,\infty)$ jumps out at us. We have

$$g(x) = \frac{\sin^2(x)}{1+x^2} e^{-x^2/2}$$

and thus

$$\int_{-\infty}^{\infty} g(x)\,dx = \int_{-\infty}^{\infty} \frac{g(x)}{f(x)} f(x)\,dx = E\left[\frac{g(X)}{f(X)}\right] = E\left[\frac{\sqrt{2\pi}\sin^2(x)}{1+x^2}\right].$$

Therefore we use a normally distributed sample for X and evaluate it at $\frac{\sqrt{2\pi}\sin^2(x)}{1+x^2}$, then average. Furthermore, we are asked to have an error of no more than $\beta = 0.01$ at the 95% confidence level. As observed in the notes, to bound the error by β with the same confidence, require that $z_{1-\alpha/2}\frac{S(n)}{\sqrt{n}} \leq \beta$. We experimented with the following script until we found a sample size N that worked. Here we used quad to integrate from -10 to 10, since outside this interval, the function is at most $1e-24$, to provide us with a good estimate of the true value of the integral.

Note: an alternative is to confine attention to a finite interval outside which the integrand is negligible, e.g., $[-6,6]$, and to use $f(x) = \frac{1}{12}$ on that interval, which is the p.d.f. for a uniformly distributed r.v. on $[-6,6]$ whose variates can be generated by the command

X = -6 + 12*rand(N,1);

One would then sample the r.v. $\frac{g(X)}{f(X)} = 12g(X)$, calculate its mean, variance and confidence intervals.

Sample output is listed below. We found $N = 3900$ to be quite adequate.

The script:

```
% script:  finalexer_7.m
disp('Final Exam Exercise 7')
disp('Sample Mean with Unmodified Variates:')
g = @(x) sin(x).^2.*exp(-x.^2/2)./(1+x.^2);
gf = @(x) sqrt(2*pi)*sin(x).^2./(1+x.^2);
truevalue = quad(g,-10,10)
alpha = 0.05; % (1-alpha) confidence level
z = stdn_inv(1-alpha/2);
randn('seed',0) % reset the counter
N = 4000
X = gf(randn(N,1));
% get the mean and variance
mu = mean(X);
sigma2 = var(X);
disp('We require that this number be smaller than bta = 0.01:')
H = z*sqrt(sigma2/N)
disp('which gives us a confidence interval of:')
mu - H, mu + H
disp('Actual error is:')
abs (mu - truevalue)
```

The output:

```
finalexer_7
Final Exam Exercise 7
Sample Mean with Unmodified Variates:
truevalue =
0.5011
N =
4000
```

```
We require that this number be smaller than bta = 0.01:
H =
0.0099
which gives us a confidence interval of:
ans =
0.4818
ans =
0.5015
Actual error is:
ans =
0.0094
diary off
```