Numerical Analysis for the Black-Scholes Model

References:

1. (text) P. Wilmott, S. Howison and J. Dewynne, The Mathematics of Financial Derivatives (Ch. 5, 7, 9), Cambridge University Press (1995).

2. C. Elliot and J. Ockendon, Weak and Variational Methods for Moving Boundary Problems, Pittman Publishing (1982).

3. R. White, An Introduction to the Finite Element Methods with Applications to Nonlinear Problems, John Wiley and Sons (1985).

4. T. Shores, Numerical Partial Differential Equations: An Introduction, lecture notes available in Public directory (2003).

OUTLINE OF THESE LECTURES:

1. Black-Sholes Model for American Options
   (a) Overview
   (b) Transformations
2. Variational Inequalities
   (a) Elliptic Problems
   (b) Parabolic Problems
3. Numerical Solutions to Variational Inequalities
   (a) Elliptic Problems
   (b) Numerical ODEs
   (c) Parabolic Problems
4. Numerical Experiments

1: Black-Scholes Model for American Options

(a) Overview.

With notation

- S: asset price;
- t: time;
- $V(S,t)$: value of an option $V = C$ for a call and $V = P$ for a put;
- σ: volatility of underlying asset;
- E: exercise price;
- T: expiry time;

Black-Scholes for American option reduces to existence of an asset price $S_f(t)$ for each time t (a "free boundary") such that the option should not be exercised on one side of $S_f(t)$ and should on the other side, so that $S_f(t)$ is an optimal price. Specifically, set $F = S - E$ if the option is a call and $F = E - S$ if the option is a put. Then $V(S,t)$ must always satisfy the constraint

$$V(S,t) \geq \max\{F, 0\}.$$

Where the constraint is an equality, we have a Black-Scholes *inequality*

$$\frac{\partial V}{\partial t} + \frac{1}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV < 0$$

and where the constraint is strict, we have the Black-Scholes *equality*

$$\frac{\partial V}{\partial t} + \frac{1}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

We also have the usual boundary and initial conditions of the calls and puts.

About calls...

Does the free boundary even exist? If not, the option should not be exercised immediately (arbitrage), so it is optimal to wait until the expiry date to exercise it and the American option reduces to an European option. We have:

- American put: yes, there is a free boundary.
- American call with constant dividend $0 < D_0 (< r)$: yes.
- American call without dividends: no.

In the case that there is a free boundary, we need extra conditions at the free boundary that help to specify it, namely that $V(S,t)$ is smooth there with

$$\frac{\partial V}{\partial S}(S_f(t),t) = \pm 1$$

using $+$ for a call and $-$ for a put.

(b) Transformations.

To keep things simple, we'll focus on American puts, since American calls with dividend require slightly different substitutions (see text, Section 6.2.2). Recall (Section 5.4 of text) that we make the substitutions

$$S = Ee^x,\ t = T - \tau/\frac{1}{2}\sigma^2,\ P = Ee^{\alpha x + \beta\tau} u(x,\tau),$$

where

$$k = r/\frac{1}{2}\sigma^2,\ \alpha = -\frac{1}{2}(k-1),\ \beta = -\frac{1}{4}(k+1)^2$$

and

$$g(x,\tau) = e^{\frac{1}{4}(k+1)^2\tau} \max\{e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0\}.$$

One obtains the Black-Scholes equation and boundary/initial conditions in the much simpler form

$$\begin{aligned} \frac{\partial u}{\partial \tau} &= \frac{\partial^2 u}{\partial x^2}, -\infty < x < \infty,\ \tau > 0, \\ u(x,0) &= g(x,0), -\infty < x < \infty, \\ u(-\infty,\tau) &= g(\infty,\tau), \tau > 0, \\ u(\infty,\tau) &= g(\infty,\tau), \tau > 0. \end{aligned}$$

In practice, we replace $\pm\infty$ by $\pm x_N$, where x_N is large. Also, the constraint condition transforms to

$$u(x,\tau) \geq g(x,\tau)$$

and the free boundary transforms to $x_f(\tau)$ and the smoothness condition on $P(S,t)$ carries over to $u(x,\tau)$.

Finally, where the constraint is an equality, we have a Black-Scholes *inequality*

$$-\frac{\partial u}{\partial \tau} + \frac{\partial^2 u}{\partial x^2} < 0 \text{ or } \frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} > 0$$

and where the constraint is strict, we have the equality

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} = 0.$$

Notice that this "classical problem", consisting of an inequality constraint on the smooth solution, a PDE which the solution satisfies on the region of strict inequality, boundary and initial conditions, is exactly equivalent to a so-called *linear complementarity problem*

$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right) \cdot (u - g) = 0,$$

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \geq 0,$$

$$u - g \geq 0.$$

The big advantage of this latter formulation is that there is no mention of the free boundary. It is computed a posteriori from the computed solution to the linear complementarity problem. Now imagine for a moment that for some strange reason we actually *knew* what $\partial u / \partial \tau$ were, say $\partial u / \partial \tau = f$. (Odd, but we have ulterior motives, so play along with it.) This is the sort of problem we want to tackle in the next section.

2: Variational Inequalities

(a) Elliptic Problems.

The problem we are going to consider is a so-called *obstacle problem*, though this entire discussion carries over to more complicated second order elliptic boundary value problems in a rather abstract setting (see White's text ref. 3) for a really nice but gentle presentation of the more abstract setting. Here's a picture of a possible scenario:

Imagine an ideal string in steady state lying over an obstacle with tension T and a pressure function $p(x)$, $0 \leq x \leq 1$. Let $f(x) = p(x)/T$. Let $u(x)$ be the vertical displacement of the string and $g(x)$ the graph of the smooth obstacle. Suppose the string is fastened at height zero on both sides away from the obstacle, say at points $(0, 0)$ and $(1, 0)$. Let D be the set of points in $(0, 1)$ at which the string makes contact with the obstacle. We also suppose that the obstacle is sufficiently convex at points of contact, that is, $-g_{xx} \geq f(x)$ for $x \in D$. For $f(x) \leq 0$ (downward pressure) this is a vacuous hypothesis. We also want to allow for the possibility that there is a force proportional to the displacement acting on the string as well, say $-cu$, where $c \geq 0$.

Classical Formulation: Find a function $u(x)$ with continuous second derivative that satisfies

$$\begin{aligned} -u_{xx} \;\; +cu &= \;\; f \;\; \text{on } [0,1]\backslash D, \\ u \;\; &= \;\; g \;\; \text{on } D, \\ -u_{xx} \;\; +cu &\geq \;\; f \;\; \text{on } [0,1], \\ u \;\; &\geq \;\; g \;\; \text{on } [0,1], \\ u(0) \;\; &= \;\; 0, \\ u(1) \;\; &= \;\; 0. \end{aligned}$$

Note the homogeneous boundary conditions, which we would not have in the Black-Scholes context. Non-homogeneous boundary conditions are fairly easy to handle, but we exclude them for convenience. The theorem we are going to develop holds for non-homogeneous conditions as well.

We won't give a formal definition of elliptic problems, but these are time independent problems steady state boundary value which include equations like $-u_{xx} = f$, in the case of one space variable.

Energy Formulation: Define a (closed, convex) set K consisting of test functions smooth and satisfying $v(x) \geq g(x)$ on $[0, 1]$, and satisfying the boundary conditions $v(0) = 0 = v(1)$. The problem is to find a $u(x) \in K$ that minimizes the energy functional

$$E(v) = \frac{1}{2} \int_0^1 \left(v_x^2 + cv^2 - 2vf \right) dx$$

over all $v \in K$. Such a function is an *energy solution*.

Variational Form: Find a $u \in K$ such that the *elliptic variational inequality* (EVI)

$$\int_0^1 \left(u_x \left(v - u \right)_x + u \left(v - u \right) \right) dx \geq \int_0^1 f \cdot \left(v - u \right) dx$$

holds for all $v \in K$. Such a u is called a *variational inequality solution*.

What are the connections between these forms of the problem, if any?

Theorem 1. Let K be defined as above.

(a) Energy solutions are variational inequality solutions.

(b) Variational inequality solutions are energy solutions.

(c) Classical solutions are variational inequality solutions.

(d) Variational inequality solutions are unique.

(e) Variational inequality solutions exist.

Remark. This theorem is very useful for a number of reasons.

(1) It shows that if there is a classical solution there is only one.

(2) The concepts of energy and variational inequality solutions enlarge the scope of problems that we can handle. Note that we no longer require that $u(x)$ have a second derivative. As a matter of fact, it doesn't even have to have a first derivative in the usual sense – rather a "weak" derivative.

(3) The energy form is going to give us an easy way to develop a numerical scheme for solving this problem.

(b) Parabolic Problems

What we are mainly interested in is the following problem: Let D be the set of all points in the xt-plane such that $u(x,t) = g(x,t)$ and let sets $K(t)$ consisting of smooth test functions satisfying $v(x,t) \geq g(x,t)$ on $[0,1] \times [0,T]$, and satisfying the boundary conditions $v(0,t) = 0 = v(1,t)$.

Classical Formulation: Find a function $u(x,t)$ with smooth second partials that satisfies

$$\begin{aligned} u_t - u_{xx} &= f \quad \text{on } [0,1] \times [0,T] \backslash D, \\ u &= g \quad \text{on } D, \\ u_t - u_{xx} &\geq f \quad \text{on } [0,1] \times [0,T], \\ u &\geq g \quad \text{on } [0,1] \times [0,T], \\ u(0,t) &= 0, \\ u(1,t) &= 0, \\ u(x,0) &= u_0(x) \in K(0) \end{aligned}$$

Variational Form: Find a function $u(x,t)$ such that $u(\cdot,t) \in K(t)$ for $t \in [0,T]$ and u_t is defined and integrable in a certain sense such that the parabolic *variational inequality* (PVI)

$$\int_0^1 u_t\,(v-u)\,dx + \int_0^1 u_x\,(v-u)_x\,dx \geq \int_0^1 f\,(v-u)\,dx$$

holds for all $v \in K(t)$ and "almost all" t and Such a u is called a *variational inequality solution*.

What are the connections between these forms of the problem, if any?

Theorem 2. Let $K(t)$ be defined as above.

(a) Classical solutions are variational inequality solutions.

(b) Variational inequality solutions are unique.

(c) Variational inequality solutions exist.

These theorems are rather more subtle. See Ref. 2 for a careful presentation of this theorem and other results.

3: Numerical Solutions to Variational Inequalities

(a) Elliptic Variational Inequalities

We are going to consider the possibility that test functions take values other than zero. So we redefine a (closed, convex) set K consisting of test functions smooth and satisfying $v(x) \geq g(x)$ on $[0,1]$, and satisfying the boundary conditions $v(0) = g(0)$ and $v(1) = g(1)$. The problem is to find a $u(x) \in K$ that minimizes the energy functional

$$E(v) = \frac{1}{2}\int_0^1 \left(v_x^2 + cv^2 - 2vf\right) dx$$

over all $v \in K$.

We are going to approximate this infinite dimensional problem by a finite dimensional one that we can plug into a computer.

Notations:

N: number of unknown nodes.

$h = \dfrac{1}{N+1}$: number of subintervals (step size).

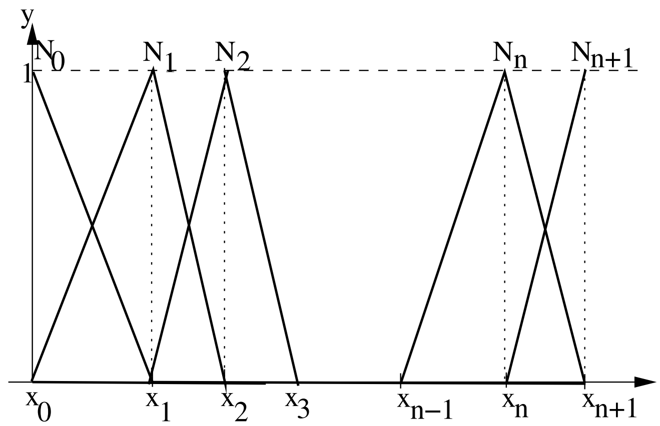
$x_j = j \cdot h$: jth node, $j = 0, \ldots, N+1$.

$k_j = k(x_j)$: for a given function $k(x) \in C[0,1]$.

K_h: the set of all $v(x) \in C[0,1]$ such that $v_j \geq g_j$ and $v(x)$ is linear on the subintervals $[x_j, x_{j+1}]$, $j = 1, \ldots, N$.

$N_i(x)$: the piecewise linear function (*hat functions, chapeau functions, cardinal functions*) on subintervals $[x_j, x_{j+1}]$ such that

$$N_i(x_j) = \begin{cases} 1, & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}.$$



Hat Functions on [0,1]: $x_0 = 0$, $x_1 = 1$.

Key Fact: Any continuous function $k(x)$ that is linear on subintervals $[x_j, x_{j+1}]$, $j = 0, \ldots, N$, is uniquely represented by the formula

$$k(x) = \sum_{j=0}^{N+1} k(x_j) N_j(x).$$

The reason is easily seen: linear combinations of continuous functions linear on the subintervals have the same property. So the right hand side is continuous and piecewise linear. Moreover, it agrees with $k(x)$ at each of the nodes x_j, $j = 0, \ldots, N+1$. Since linear functions are uniquely determined by two values, the result follows.

If $k(x)$ is not piecewise linear, the above formula gives the so-called *linear interpolant approximation* $k^h(x)$ to $k(x)$, that is,

$$k^h(x) = \sum_{j=0}^{N+1} k(x_j)N_j(x).$$

About derivatives...

A technical detail about hat functions is the matter of a derivative function. Obviously, one doesn't exist at bends, so $N(x)$ doesn't have a derivative in the classical sense. We *define* $N_j'(x)$ to be $1/h$ on the interval $(x_{j-1}, x_j]$, $-1/h$ on the interval $(x_j, x_{j+1}]$ and 0 otherwise. In this way we recover the fundamental theorem of calculus:

$$\int_a^b N_j'(x)dx = N(b) - N(a)$$

for any $0 \leq a \leq b \leq 1$.

We can now formulate the discrete model for the Energy Formulation. Approximate the solution $v(x)$, right hand side $f(x)$ and barrier function $g(x)$ by their linear interpolants. Observe that the first and last values of $v(x)$, v_0 and v_{N+1}, are known, while the values $v_1, v_2, \ldots, v_N$ are unknown.

Introduce notation for the known (in principle) constants

$$a_{jk} = \int_0^1 \left(N_j'(x) N_k'(x) + c N_j(x) N_k(x) \right) dx,$$

$$b_j = \int_0^1 N_j(x) f^h(x) dx.$$

Next, calculate the energy functional

$$
\begin{aligned}
E(v^h) &= \frac{1}{2}\int_0^1 \left(\left(v_x^h\right)^2 + c\left(v^h\right)^2 - 2v^h f^h\right) dx \\
&= \frac{1}{2}\int_0^1 \left(\sum_{j=0}^{N+1} v_j N_j'(x)\right)\left(\sum_{k=0}^{N+1} v_k N_k'(x)\right) dx \\
&\quad + \frac{1}{2}\int_0^1 c\left(\sum_{j=0}^{N+1} v_j N_j(x)\right)\left(\sum_{k=0}^{N+1} v_k N_k(x)\right) dx \\
&\quad - \int_0^1 \sum_{j=0}^{N+1} v_j N_j(x) f^h(x) dx \\
&= \frac{1}{2}\sum_{j=0}^{N+1}\sum_{k=0}^{N+1} v_j a_{jk} v_k - \sum_{j=0}^{N+1} v_j b_j
\end{aligned}
$$

This is just a quadratic function in the variables $v_1, v_2, \ldots, v_N$. As we noted before, v_0 and v_{N+1} are not variables – they are known values $v(0)$ and $v(1)$.

For the moment, let's ignore the constraints on the v_j's. As in calculus, we would differentiate the energy functional with respect to $v_1, \ldots, v_N$ and set the results equal to zero to locate the critical points. Use the symmetry property $a_{jk} = a_{kj}$ and a little calculation shows that

$$\sum_{k=0}^{N+1} a_{mk} v_k - b_m = 0,\ m = 1, 2, \ldots, N.$$

Now let the $N \times N$ matrix A have entries a_{jk}, the $N \times 1$ vector $\mathbf{b}$ have entries $b_1 - a_{01} v_0, b_2, \ldots, b_N - a_{N,N+1} v_{N+1}$ and the $N \times 1$ vector $\mathbf{v}$ have entries $v_1, v_2, \ldots, v_N$. Then the problem reduces to the simple linear system

$$A\mathbf{v} = \mathbf{b}.$$

Now it can be proved that the matrix A is symmetric positive definite, hence non-singular. Therefore, this system has a unique solution.

We could find the solution $\mathbf{v}$ by Gaussian elimination, as in Math 314. However, there is another approach that we want to explore because it can be extended to handle the constraints of this problem.

Gauss-Seidel-SOR Method:

The idea is to iteratively approximate a solution to $A\mathbf{v} = \mathbf{b}$, starting from some initial guess $\mathbf{v}^0$ For example, we could use the first equation to solve for v_1^1 in terms of $v_2^0, \ldots, v_N^0$, then use the second equation to solve for v_2^1 in terms of $v_1^1 v_3^0, \ldots, v_N^0$, etc. In equation form

$$v_j^{n+1} = \left(b_j - \sum_{k<j} a_{jk} v_k^{n+1} - \sum_{k>j} a_{jk} v_k^n \right) / a_{jj},\ j = 1, 2, \ldots$$

where the iteration is repeated for $n = 0, 1, \ldots$ until convergence. However, one can think of the term above as a "relaxation term" in the sense that, with $\omega = 1$,

$$v_j^{n+1} = v_j^n + \left(v_j^{n+1} - v_j^n \right) = v_j^n + \omega \left(v_j^{n+1} - v_j^n \right).$$

This suggests that we could "overrelax" the approximation by using larger ω. Hence, the SOR method:

$$
\begin{aligned}
\widehat{v}_j^{n+1} &= \left(b_j - \sum_{k<j} a_{jk} v_k^{n+1} - \sum_{k>j} a_{jk} v_k^n\right) / a_{jj} \\
v_j^{n+1} &= v_j^n + \omega\left(\widehat{v}_j^{n+1} - v_j^n\right) = (1-\omega)\, v_j^n + \omega \widehat{v}_j^{n+1}
\end{aligned}
$$

Theorem 3. If A is a symmetric positive definite matrix and $0 < \omega < 2$, then the Gauss-Seidel-SOR method converges from any starting guess. Moreover, there is an optimal $\omega = \omega^*$ (most rapid convergence rate), where $1 < \omega^* < 2$.

Back to our problem, where we now pay attention to the constraints on the v_j's. If we take the point of view that the jth equation is used to solve for v_j, then we must have either $v_j > g_j$ or the jth equation holds.

One could use “linear complementarity” to express this condition. We simply observe that a small adjustment in SOR enables us to respect these constraints.

Theorem 4. Suppose that A is a symmetric positive definite $N \times N$ matrix, $0 < \omega < 2$ and $\mathbf{g}$ and $\mathbf{v}^0$ are $N \times 1$ vectors and

$$K = \{\mathbf{v} \,|\, \mathbf{v} \text{ is } N \times 1 \text{ and } v_j \geq g_j,\ j = 1, \ldots, N\}.$$

Then the *projected SOR method*

$$\begin{aligned} \widehat{v}_j^{n+1} &= \left(b_j - \sum_{k<j} a_{jk} v_k^{n+1} - \sum_{k>j} a_{jk} v_k^n \right) / a_{jj} \\ v_j^{n+1} &= \max\left\{ g_j, (1-\omega)\, v_j^n + \omega \widehat{v}_j^{n+1} \right\} \end{aligned}$$

generates a sequence of vectors $\{\mathbf{v}^n\}_{n=0}^{\infty}$ that converges to the solution to the problem of minimizing

$$E(\mathbf{v}) = \min_{\mathbf{v} \in K} \left\{ \mathbf{v}^T A \mathbf{v} - \mathbf{v}^T \mathbf{b} \right\}.$$

This pretty well completes the description of our numerical algorithm. A few more mechanical details are needed.

About integrals...

These formulas are easily checked directly:

$$\int_0^1 N_j'(x)N_k'(x)dx = \begin{cases} 0, & |j-k|>1 \\ -1/h, & k=j\pm 1 \\ 2/h, & 0<k=j<N+1 \\ 1/h, & k=j=0 \text{ or } N+1 \end{cases}$$

$$\int_0^1 N_j(x)N_k(x)dx = \begin{cases} 0, & |j-k|>1 \\ h/6, & k=j\pm 1 \\ 2h/3, & 0<k=j<N+1 \\ h/3, & k=j=0 \text{ or } N+1 \end{cases}$$

$$\int_0^1 N_j(x)f^h(x)dx = \begin{cases} \frac{h}{6}\left(2f_0+f_1\right), & j=0 \\ \frac{h}{6}\left(f_{j-1}+4f_j+f_{j+1}\right), & 0<j<N+1 \\ \frac{h}{6}\left(f_N+2f_{N+1}\right), & j=N+1 \end{cases}$$

(b) Numerical ODEs

Well, not really. Actually, the question we have to deal with is simpler, but the connection to ODEs is that one uses the Taylor series kinds of argument we're about to see in an essential way in analyzing the merit of various numerical ODE methods.

Question: How do we approximate a function dy/dt given values of $y(t)$ at various node points?

For purpose of simplicity, let's say that we have equally spaced nodes in steps of k. Let $t_j = j$ and $y_j = y(jk)$. Recall the Taylor formulas

$$y(t_{j+1}) = y(t_j) + y'(t_j)k + y''(t_j)\frac{k^2}{2} + \mathcal{O}(k^3)$$

$$y(t_{j-1}) = y(t_j) - y'(t_j)k + y''(t_j)\frac{k^2}{2} + \mathcal{O}(k^3)$$

Use the first to solve for $y'_j = y'(t_j)$ and obtain the *forward difference formula*

$$y'_j = \frac{y_{j+1} - y_j}{k} + \mathcal{O}(k).$$

Similarly, the second formula gives the *backward difference formula*

$$y'_j = \frac{y_j - y_{j-1}}{k} + \mathcal{O}(k).$$

Next, we subtract the second Taylor formula from the first and solve for y'_j to obtain

$$y'_j = \frac{y_{j+1} - y_{j-1}}{2k} + \mathcal{O}(k^2).$$

This is the so-called *centered difference formula*. These order formulas suggest that as k gets smaller, the centered difference formula should exceed the others in accuracy. If we add these two Taylor formulas together, we also get something rather interesting fact about averages:

$$y_j = \frac{y_{j-1} + y_{j+1}}{2} + \mathcal{O}(k^2).$$

One more point along these lines: wouldn't it be nice if somehow the simple forward differences, which have the merit of only using values at two adjacent nodes, were second order accurate? Well, if you squint, they are! Suppose that we had written out the Taylor formulas above for half steps $k/2$. What we would have ended up with is

$$y'_{j+1/2} = \frac{y_{j+1} - y_j}{k} + \mathcal{O}\left(\left(\frac{k}{2}\right)^2\right) = \frac{y_{j+1} - y_j}{k} + \mathcal{O}(k^2)$$

where we understand that $y_{j+1/2} = y(t_{j+1/2}) = y((j+1/2)k)$. This is a so-called value of y at a "half-node". If we also wanted a second order accurate value for y itself at this half-node, then we would use the averaging formula to obtain that

$$y_{j+1/2} = \frac{y_{j+1} + y_j}{2} + \mathcal{O}(k^2).$$

(c) Parabolic Variational Inequalities

We're now going to combine parts (a) and (b) to tackle the parabolic variational inequality.

4: Numerical Experiments