# Additive Codes Associated to Laplacian Simplices


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*Joint with Marie Meyer

# Outline

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1. (Ehrhart) Theory of simplices
2. Laplacian simplices
3. Reflexive Laplacian simplices, codes, and duality

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# (Ehrhart) Theory of simplices

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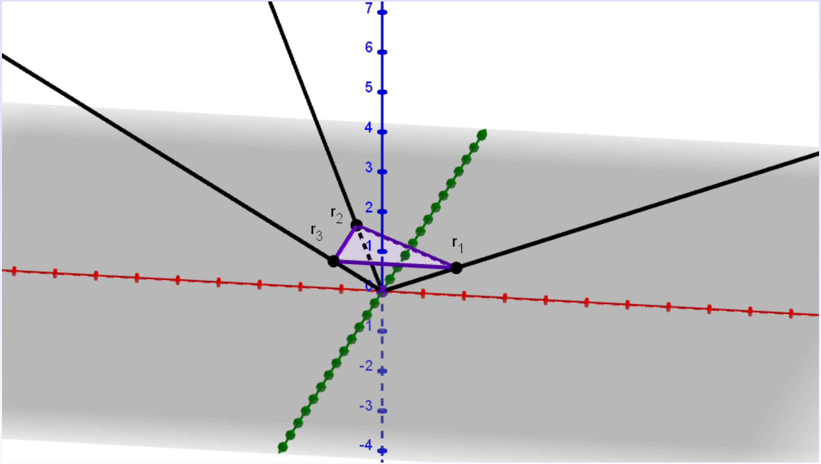

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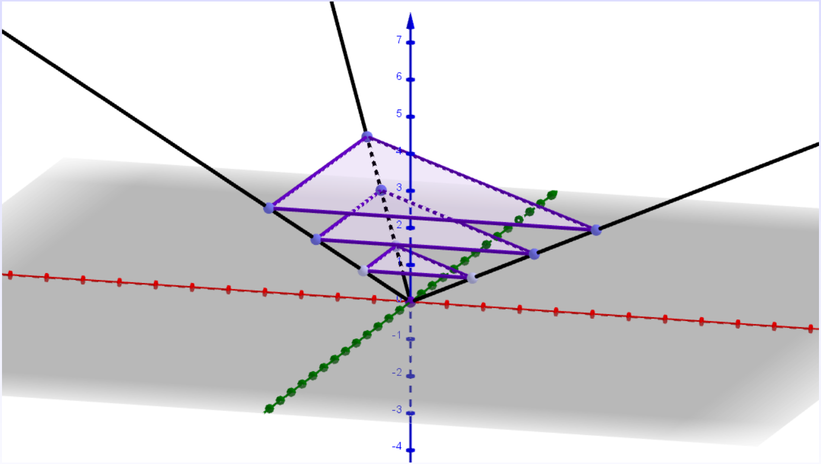

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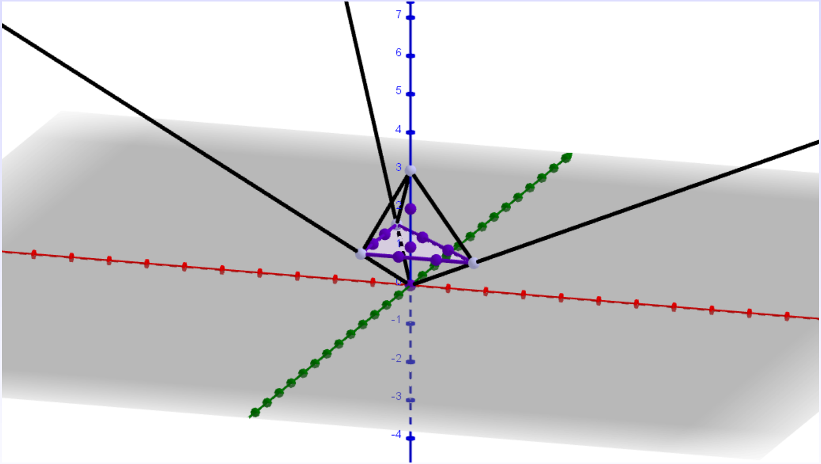

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- The $h^*$**-vector** of $\Delta$ is $h^*(\Delta) = (h_0, h_1, \ldots, h_d)$ where
$$h_i = \#\{\mathbf{p} \in \Pi(\Delta) \cap \mathbb{Z}^{d+1} \mid \mathbf{p}_{d+1} = i\}.$$

# (Ehrhart) Theory of simplices: A different approach

- $$\Lambda(\Delta) := \left\{ \lambda = (\lambda_1, \ldots, \lambda_{d+1}) \;\middle|\; \sum_{i=1}^{d+1} \lambda_i (\mathbf{v}_i, 1) \in \Pi(\Delta) \cap \mathbb{Z}^{d+1} \right\}.$$

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  $\mathrm{ht}(\lambda) := \sum_{j=1}^{d+1} \lambda_j$ is called the **height** of $\lambda$.

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- Let $G$ be a simple connected graph with $n$ vertices. Denote $L_G$ its Laplacian matrix and $\tau(G)$ the number of spanning trees.

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**Definition (Braun/Meyer, 2017)**

The convex hull of the rows of $L_G(n)$, denoted $\Delta_G$, is called the **Laplacian simplex associated to $G$**.

# Laplacian simplices

- Let $A \in \mathbb{Z}^{n \times n}$ be a square matrix. View $A$ as the $\mathbb{Z}$-module homomorphism $A : \mathbb{Z}_m^n \to \mathbb{Z}_m^n$, $x \mapsto xA$.

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**Theorem (Braun/Meyer, 2017)**

*Let $G$ be a simple connected graph on $n$ vertices. Then*

$$\Lambda(\Delta_G) = \left\{ \frac{\mathbf{x}}{n\tau(G)} \;\middle|\; \bar{\mathbf{x}} \in \ker_{\mathbb{Z}_{n\tau(G)}} [L(n) \mid 1] \right\}.$$

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**Definition**

Let $G$ be a simple connected graph on $n$ vertices such that $\Delta_G$ is reflexive. Then $\mathcal{C}(\Delta_G) := \ker_{\mathbb{Z}_n}[L(n) \mid 1] \subseteq \mathbb{Z}_n^n$ is called the **additive code associated to the (reflexive) Laplacian simplex** $\Delta_G$.

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**Theorem (Meyer/P, 2018)**

*Let $G$ be a simple connected graph with $n$ vertices such that the associated $\Delta_G$ is reflexive. Then*

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- **IDEA**: Use MacWilliams Duality.

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**Theorem (Meyer/P, 2018)**

*Let $a \leq b$ be any natural numbers. Then there exists a graph $G$ such that $\mathcal{C}(\Delta_G)$ has rate arbitrarily close to $a/b$.*

Thank You!