Laplacian Simplices II: A Coding Theoretic Approach

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Outline

1. (Ehrhart) Theory of simplices

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2. Laplacian simplices
3. Reflexive Laplacian simplices

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(Ehrhart) Theory of simplices

- A **simplex** Δ in $\mathbb{R}^d$ is a full-dimensional convex hull of $d+1$ points $\mathbf{v}_1, \ldots, \mathbf{v}_{d+1}$ (in $\mathbb{R}^d$).

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- (**Batyrev and Hofscheier**):

$$\Lambda(\Delta) := \left\{ \lambda = (\lambda_1, \ldots, \lambda_{d+1}) \,\middle|\, \sum_{i=1}^{d+1} \lambda_i(\mathbf{v}_i, 1) \in \Pi(\Delta) \cap \mathbb{Z}^{d+1} \right\}.$$

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- If $h^*(\Delta)$ is symmetric then Δ is called **reflexive**.

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- $\text{Vol}(\Delta_G) = n \cdot \tau(G)$.

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Reflexive Laplacian simplices

Theorem (Meyer/P, 2018)

Let G be a simple connected graph on n vertices such that Δ_G is reflexive. Then

$$\Lambda(\Delta_G) = \left\{ \frac{\mathbf{x}}{n} \,\middle|\, \overline{\mathbf{x}} \in \ker_{\mathbb{Z}_n}[L(n) \mid 1] \right\}.$$

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$$\Lambda((\Delta_G)^\vee) \cong \Lambda(\Delta_G)^\circ.$$

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4. Future research

Future research I: Unimodality

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Let G be the graph K_3 with a 7-edge path as a tail. Then

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Open Problem

Classify reflexive Laplacian simplices with unimodal h^*-vector.

Future research II: Duality

Theorem (Meyer/P, 2018)

Let T_n and K_n denote any tree on n vertices and complete graph respectively. Then

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Find pairs of graphs (on n vertices) (G, H) such that $\Lambda(\Delta_G)^\circ \cong \Lambda(\Delta_H)$.

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- Note: A necessary condition is $\tau(G) \cdot \tau(H) = n^{n-2}$.

Future research III: h^*-vector of the dual

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- Motivation: MacWilliams Duality and MacWilliams Identity.

Thank You!