

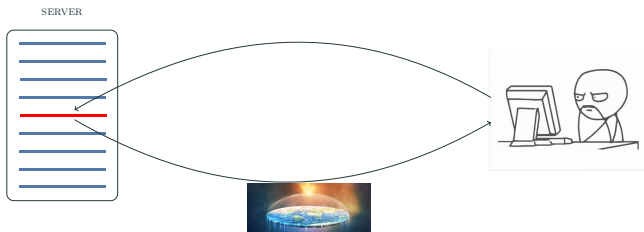
Quantum Private Information Retrieval

Private Information Retrieval with Quantum Resources

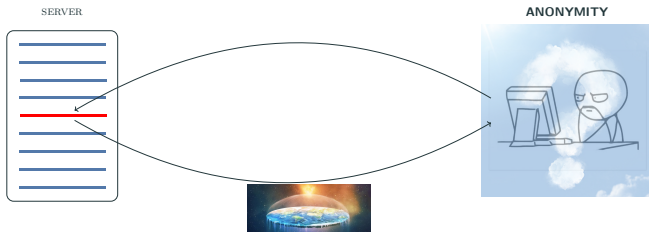
Tefjol Pllaha

Department of Mathematics
University of Nebraska - Lincoln

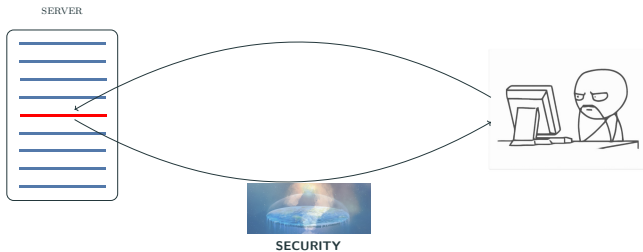
Typical Online Activity



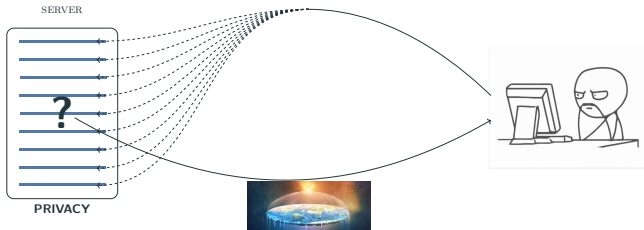
Typical Online Activity: **Anonymity**



Typical Online Activity: Security



Typical Online Activity: Privacy



Some Milestones

- Chor et al. 1995: Seminal paper on PIR.
 - Privacy **only** achievable by downloading the entire database.
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 - Anonymization, differential privacy, data protection laws ...

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- Quest for practical solutions continues.

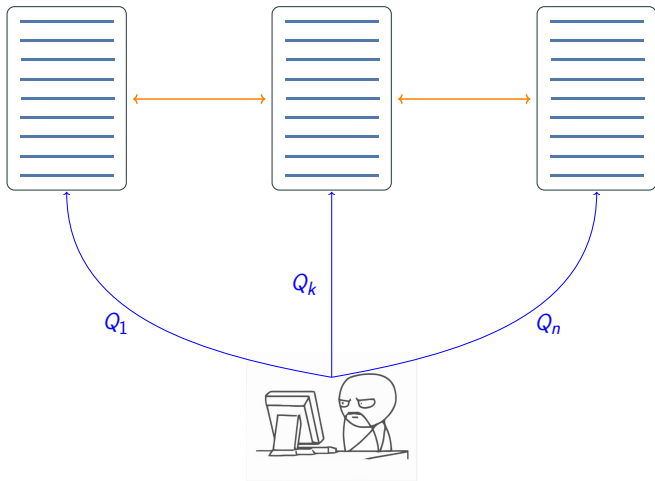
Coded Storage

m files $x^1, \dots, x^m \in \mathbb{F}_q^{\beta \times k}$ are encoded and stored on n servers by a $[n, k]$ storage code \mathcal{C} .

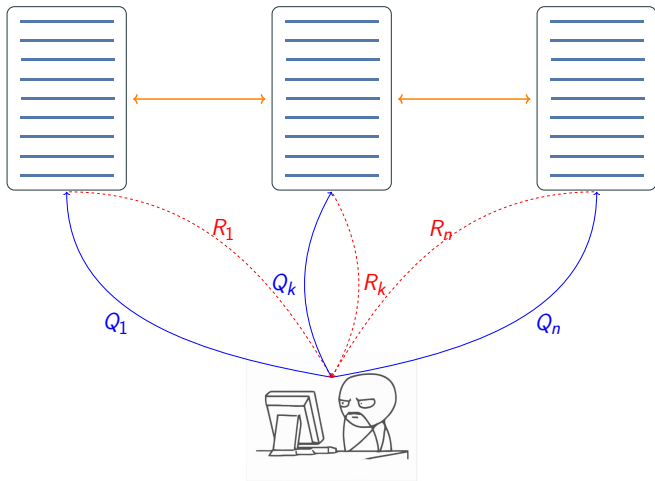
$$\begin{array}{l} \text{file 1} \\ \vdots \\ \text{file } m \end{array} \begin{pmatrix} \boxed{\begin{matrix} x_{1,1}^1 & \cdots & x_{1,k}^1 \\ \vdots & \ddots & \vdots \\ x_{\beta,1}^1 & \cdots & x_{\beta,k}^1 \end{matrix}} \\ \vdots \\ \boxed{\begin{matrix} x_{1,1}^m & \cdots & x_{1,k}^m \\ \vdots & \ddots & \vdots \\ x_{\beta,1}^m & \cdots & x_{\beta,k}^m \end{matrix}} \end{pmatrix} \cdot \mathbf{G}_{\mathcal{C}} = \begin{pmatrix} \boxed{\begin{matrix} y_{1,1}^1 \\ \vdots \\ y_{\beta,1}^1 \end{matrix}} & \cdots & \boxed{\begin{matrix} y_{1,n}^1 \\ \vdots \\ y_{\beta,n}^1 \end{matrix}} \\ \vdots & \ddots & \vdots \\ \boxed{\begin{matrix} y_{1,1}^m \\ \vdots \\ y_{\beta,1}^m \end{matrix}} & \cdots & \boxed{\begin{matrix} y_{1,n}^m \\ \vdots \\ y_{\beta,n}^m \end{matrix}} \end{pmatrix}$$

SERVER_1 SERVER_n

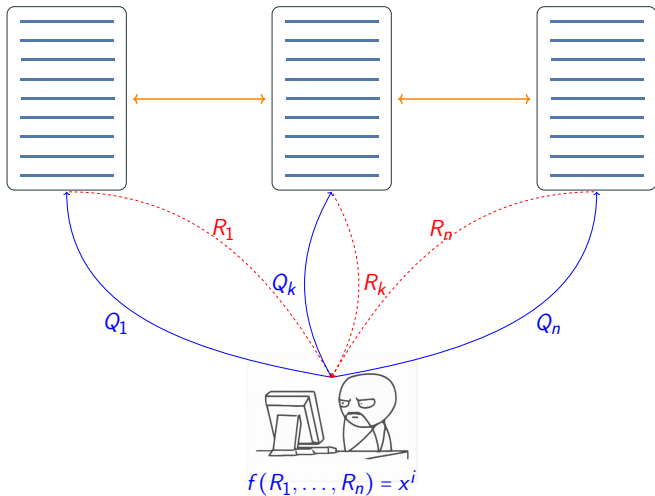
Private Information Retrieval (PIR)



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PIR with t -collusion (t -PIR)

Definition (t -PIR).

User privacy: Any set of at most t colluding nodes learns no information about the index i of the desired file, *i.e.*, the mutual information

$$I(i; Q_{\mathcal{T}}^K, R_{\mathcal{T}}^K, y_{\mathcal{T}}) = 0, \quad \forall \mathcal{T} \subset [n], |\mathcal{T}| \leq t .$$

Server privacy: The user does not learn any information about the files other than the requested one, *i.e.*,

$$I(x^j; Q^K, R^K, K) = 0, \quad \forall j \neq K .$$

A scheme with both user and server privacy is called *symmetric*.

PIR with t -collusion (t -PIR)

Definition (Rate and Capacity).

For a PIR scheme the **rate** is the number of information bits of the requested file retrieved per downloaded bits, *i.e.*,

$$R_{\text{PIR}} = \frac{\text{Number of bits in a file}}{\text{Number of downloaded bits}} .$$

The PIR **capacity** is the supremum of PIR rates of all possible PIR schemes, for a fixed parameter setting.

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QPIR is PIR with “*entangled servers*” and “*quantum answers*”.

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- arXiv:2001.04436, arXiv:1903.12556, arXiv:1903.10209
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- Replicated storage with $t = n - 1$ collusion.
- **Goal:** $[n, k]$ coded storage with $t = n - k$ collusion.

Ingredients for QPIR

- Quantum adaptation of existing schemes.
- Generalized Reed-Solomon codes

$$\text{GRS}_k(\alpha, v) = \{(v_i f(\alpha_i))_{1 \leq i \leq n} \mid f(x) \in \mathbb{F}_q^{<k}[x]\}.$$

- Quantum Computation.

Ingredients for QPIR: Quantum Computation

- **Bell State** $|\Phi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$.

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Ingredients for QPIR: Quantum Computation

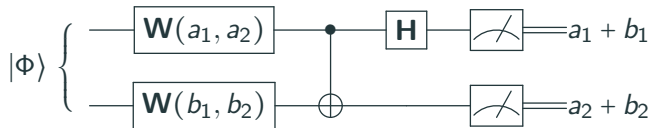
- Bell State $|\Phi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$.
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- **The PVM**

$$\mathcal{B}_{\mathbb{F}_2^2} = \{\mathbf{B}_{(a,b)} = \mathbf{W}_1(a, b)|\Phi\rangle\langle\Phi|\mathbf{W}_1(a, b)^\dagger \mid a, b \in \mathbb{F}_2\}.$$

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- **Two-Sum Protocol:** Alice and Bob send the sum $(a_1 + b_1, a_2 + b_2)$ of their bits to Carol.



A QPIR Example

- $n = 4$ servers and $[4, 2]_4$ - coded database with RS code

$$\mathbf{G}_C = \begin{pmatrix} 1 & 0 & \alpha^2 & \alpha \\ 0 & 1 & \alpha & \alpha^2 \end{pmatrix}.$$

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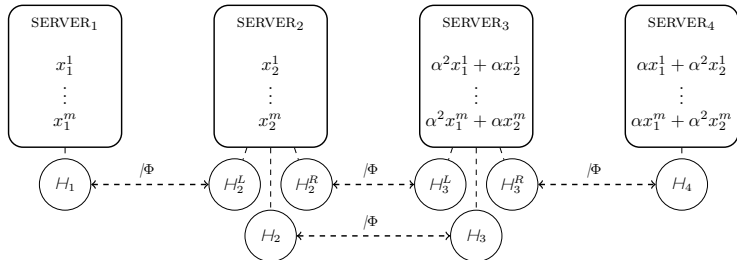
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 - k also determines the number of rounds.
- Query index K , i.e., the requested file is $x^K = (x_1^K, x_2^K)$.

A QPIR Example: Entangled Servers



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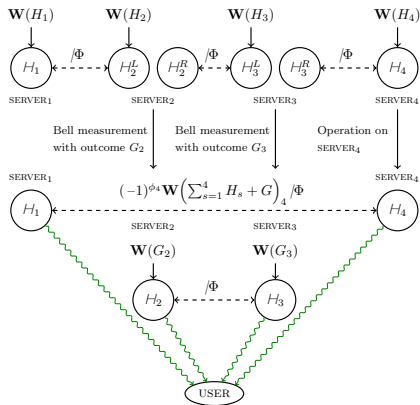
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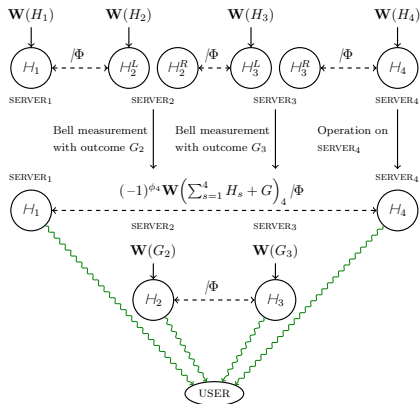
- Query Q_s to server s .

A QPIR Example: Servers' Response



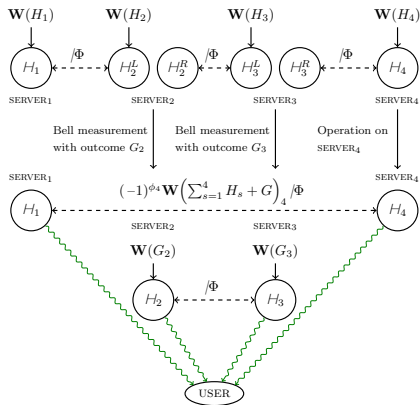
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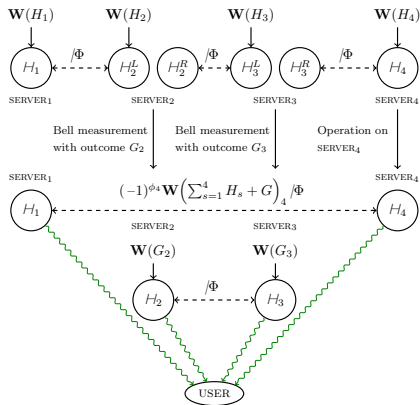
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- Each server sends its qubit to the user.

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Remark

Here we targeted servers 1&2 (systematic encoding). Since the storage is MDS-coded, one can target any two (k in general) servers.

A QPIR Example: **Secrecy, Collusion, Rate**

- **User secrecy**: queries Q_1, \dots, Q_4 independent of the index K , two random vectors generated and encoded into queries \Rightarrow at least three servers needed in order to retrieve the file requested \Rightarrow **2-collusion**.

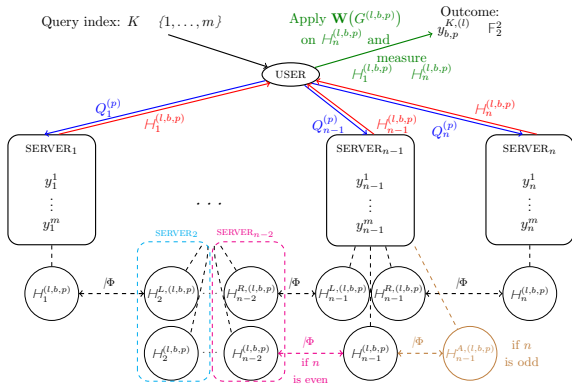
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- **Rate:** $R = \frac{2 \cdot 2}{2 \cdot 4} = \frac{1}{2}$.

QPIR with n Servers



With $n = k + t$,

$$R_{\text{PIR}} = \begin{cases} \frac{2}{n}, & \text{if } n \text{ is even,} \\ \frac{2}{n+1}, & \text{if } n \text{ is odd,} \end{cases}$$

Ingredients for t -QPIR

Star Product PIR scheme from Freij-Hollanti et al.

- Coded **storage** with storage code \mathcal{C} .
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Adapted Star Product scheme for t -QPIR

- Weakly self-dual codes.

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Lemma

Let q be even with $q \geq n$. For any $[n, k]$ GRS code \mathcal{C} there exists an $[n, t]$ GRS code \mathcal{D} such that their star-product $\mathcal{S} = \mathcal{C} \star \mathcal{D}$ is an $[n, k + t - 1]$ weakly self-dual GRS code.

Capacity [1]

CAPACITIES	PIR	ref.	SPIR	ref.	QPIR	ref.
Replicated storage, no collusion	$1 - \frac{1}{n}$	[3]	$1 - \frac{1}{n}$	[6]	1	[21]
Replicated storage, t-collusion	$1 - \frac{t}{n}$	[4]	$1 - \frac{t}{n}$	[25]	$\min\{1, \frac{2(n-t)}{n}\}$	[23]
$[n, k]$ -MDS coded storage, no collusion	$1 - \frac{k}{n}$	[5]	$1 - \frac{k}{n}$	[7]	$\min\{1, \frac{2(n-k)}{n}\}$	-
$[n, k]$ -MDS coded storage, t-collusion	$1 - \frac{k+t-1}{n}$	[12]	$1 - \frac{k+t-1}{n}$	[7]	$\min\{1, \frac{2(n-k-t+1)}{n}\}$	-

What's next?

QSDMM: **Q**uantum **S**ecure **D**istributed **M**atrix **M**ultiplication

References:

- M. Allaix, S. Song, L. Holzbaur, **T. Pllaha**, M. Hayashi, and C. Hollanti. "On the capacity of quantum private information retrieval from MDS-coded and colluding servers," *IEEE Journal on Selected Areas in Communications*, vol. 40, no. 3, pp. 885-898, March 2022.
- M. Allaix, L. Holzbaur, **T. Pllaha**, C. Hollanti. "High-Rate Quantum Private Information Retrieval with Weakly Self-Dual Star Product Codes," *In 2021 IEEE International Symposium on Information Theory*, 1046-1051.
- M. Allaix, L. Holzbaur, **T. Pllaha**, C. Hollanti. "Quantum Private Information Retrieval from Coded and Colluding Servers," *IEEE Journal on Selected Areas in Information Theory*, 1(2), 599-610, August 2020.
- M. Allaix, L. Holzbaur, **T. Pllaha**, C. Hollanti. "Quantum Private Information Retrieval from MDS-coded and Colluding Servers," *In 2020 IEEE International Symposium on Information Theory*, 1059-1064.

Thank You!