LOCAL COHOMOLOGY MODULES WITH INFINITE DIMENSIONAL SOCLES

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Abstract. In this paper we prove the following generalization of a result of Hartshorne: Let T be a commutative Noetherian local ring of dimension at least two, $R = T[x_1, \ldots, x_n]$, and $I = (x_1, \ldots, x_n)$. Let f be a homogeneous element of R such that the coefficients of f form a system of parameters for T. Then the socle of $H^n_I(R/fR)$ is infinite dimensional.

1. Introduction

The third of Huneke's four problems in local cohomology [Hu] is to determine when $H^i_I(M)$ is Artinian for a given ideal I of a commutative Noetherian local ring R and finitely generated R-module M. An R-module N is Artinian if and only $\operatorname{Supp}_R N \subseteq \{m\}$ and $\operatorname{Hom}_R(R/m, N)$ is finitely generated, where m is the maximal ideal of R. Thus, Huneke's problem may be separated into two subproblems:

- When is $\operatorname{Supp}_R H^i_I(M) \subseteq \{m\}$?
- When is $\operatorname{Hom}_R(R/m, H^i_I(M))$ finitely generated?

This article is concerned with the second question. For an R-module N, one may identify $\operatorname{Hom}_R(R/m, N)$ with the submodule $\{x \in N \mid mx = 0\}$, which is an R/m-vector space called the *socle* of N (denoted $\operatorname{soc}_R N$). It is known that if R is an unramified regular local ring then the local cohomology modules $H^i_I(R)$ have finite dimensional socles for all $i \geq 0$ and all ideals I of R ([HS], [L1], [L2]). The first example of a local cohomology module with an infinite dimensional socle was given in 1970 by Hartshorne [Ha]: Let k be a field, $R = k[[u, v]][x, y]$, $P = (u, v, x, y)R$, $I = (x, y)R$, and $f = ux + vy$. Then $\operatorname{soc}_{R_P} H^2_{IR_P}(R_P/fR_P)$ is infinite dimensional. Of course, since I and f are homogeneous, this is equivalent to saying that $\operatorname{Hom}_R(R/P, H^2_I(R/fR))$ (the **socle* of $H^2_I(R/fR)$) is infinite dimensional. Hartshorne proved this by exhibiting an infinite set of linearly independent elements in the *socle of $H^2_I(R)$.

In the last 30 years there have been few results in the literature which explain or generalize Harthshorne's example. For affine semigroup rings, a remarkable result proved by Helm and Miller [HM] gives necessary and sufficient conditions

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(on the semigroup) for the ring to possess a local cohomology module (of a finitely generated module) having infinite dimensional socle. Beyond that work, however, little has been done.

In this paper we prove the following:

Theorem 1.1. *Let (T, m) be a Noetherian local of dimension at least two. Let $R = T[x_1, \ldots, x_n]$ be a polynomial ring in n variables over T, $I = (x_1, \ldots, x_n)$, and $f \in R$ a homogeneous polynomial whose coefficients form a system of parameters for T. Then the* **socle of $H^n_I(R/fR)$ is infinite dimensional.*

Hartshorne's example is obtained by letting $T = k[[u, v]]$, $n = 2$, and $f = ux+vy$ (homogeneous of degree 1). Note, however, that we do not require the coefficient ring to be regular, or even Cohen-Macaulay. As a further illustration, consider the following:

Example 1.2. Let $R = k[[u^4, u^3v, uv^3, v^4]][x, y, z]$, $I = (x, y, z)R$, and $f = u^4x^2 + v^8yz$. Then the *socle of $H^3_I(R/fR)$ is infinite dimensional.

Part of the proof of Theorem 1.1 was inspired by the recent work of Katzman [Ka] where information on the graded pieces of $H^n_I(R/fR)$ is obtained by examining matrices of a particular form. We apply this technique in the proof of Lemma 2.8.

Throughout all rings are assumed to be commutative with identity. The reader should consult [Mat] or [BH] for any unexplained terms or notation and [BS] for the basic properties of local cohomology.

2. The Main Result

Let $R = \oplus R_\ell$ be a Noetherian ring graded by the nonnegative integers. Assume R_0 is local and let P be the homogeneous maximal ideal of R. Given a finitely generated graded R-module M we define the **socle* of M by

$$\begin{aligned} {}^*\operatorname{soc}_R M &= \{x \in M \mid Px = 0\} \\ &\cong \operatorname{Hom}_R(R/P, M). \end{aligned}$$

Clearly, $^*\operatorname{soc}_R M \cong \operatorname{soc}_{R_P} M_P$. An interesting special case of Huneke's third problem is the following:

Question 2.1. *Let $n := \mu_R(R_+/PR_+)$, the minimal number of generators of R_+. When is $^*\operatorname{soc} H^n_{R_+}(R)$ finitely generated?*

For $i \in \mathbb{N}$ it is well known that $H^i_{R+}(R)$ is a graded R- module, each graded piece $H^i_{R_+}(R)_\ell$ is a finitely generated R_0-module, and $H^i_{R_+}(R)_\ell = 0$ for all sufficiently large integers ℓ ([BS, 15.1.5]). If we know *a priori* that $H^n_{R+}(R)_\ell$ has finite length for all ℓ (e.g., if $\operatorname{Supp}_R H^n_{R_+}(R) \subseteq \{P\}$), then Question 2.1 is equivalent to:

Question 2.2. *When is $\operatorname{Hom}_R(R/R_+, H^n_{R_+}(R))$ finitely generated?*

We give a partial answer to these questions for hypersurfaces. For the remainder of this section we adopt the following notation: Let (T, m) be a local ring of dimension d and $R = T[x_1, \dots, x_n]$ a polynomial ring in n variables over T. We endow R with an $\mathbb{N}$-grading by setting $\deg T = 0$ and $\deg x_i = 1$ for all i. Let $I = R_+ = (x_1, \dots, x_n)R$ and $P = m + I$ the homogeneous maximal ideal of R. Let $f \in R$ be a homogeneous element of degree p and C_f the ideal of T generated by the nonzero coefficients of f.

Our main result is the following:

Theorem 2.3. *Assume $d \geq 2$ and the (nonzero) coefficients of f form a system of parameters for T. Then ${}^*\operatorname{soc}_R H_I^n(R/fR)$ is not finitely generated.*

The proof of this theorem will be given in a series of lemmas below. Before proceeding with the proof we make a couple of remarks:

Remark 2.4. (a) If $d \leq 1$ in Theorem 2.3 then ${}^*\operatorname{soc} H_I^n(R/fR)$ is finitely generated. This follows from [DM, Corollary 2] since $\dim R/I = \dim T \leq 1$.

(b) The hypothesis that the nonzero coefficients of f form a system of parameters for T is stronger than our proof requires. One only needs that C_f be m-primary and that there exists a dimension 2 ideal containing all but two of the coefficients of f. (See the proof of Lemma 2.8.)

The following lemma identifies the support of $H_I^n(R/fR)$ for a homogeneous element $f \in R$. This lemma also follows from a much more general result recently proved by Katzman and Sharp [KS, Theorem 1.5].

Lemma 2.5. *Let $f \in R$ be a homogeneous element. Then*

$$\operatorname{Supp}_R H_I^n(R/fR) = \{Q \in \operatorname{Spec} R \mid Q \supseteq I + C_f\}.$$

Proof: It is enough to prove that $H_I^n(R/fR) = 0$ if and only if $C_f = T$. As $H_I^n(R/fR)_k$ is a finitely generated T-module for all k, we have by Nakayama that $H_I^n(R/fR) = 0$ if and only if $H_I^n(R/fR) \otimes_T T/m = 0$. Now

$$\begin{aligned} H_I^n(R/fR) \otimes_T T/m &\cong H_I^n(R/fR \otimes_T T/m) \\ &\cong H_N^n(S/fS) \end{aligned}$$

where $S = (T/m)[x_1, \dots, x_n]$ is a polynomial ring in n variables over a field and $N = (x_1, \dots, x_n)S$. As $\dim S = n$, we see that $H_N^n(S/fS) = 0$ if and only if the image of f modulo m is nonzero. Hence, $H_I^n(R/fR) = 0$ if and only if at least one coefficient of f is a unit, i.e., $C_f = T$. □

We are mainly interested in the case the coefficients of f generate an m-primary ideal:

Corollary 2.6. *Let $f \in R$ be homogeneous and suppose C_f is m-primary. Then*

$$\operatorname{Supp}_R H_I^n(R/fR) = \{P\}.$$

Our next lemma is the key technical result in the proof of Theorem 2.3.

Lemma 2.7. *Suppose $u, v \in T$ such that* $\operatorname{ht}(u,v)T = 2$. *For each integer $n \geq 1$ let M_n be the cokernel of $\phi_n : T^{n+1} \to T^n$ where ϕ_n is represented by the matrix*

$$A_n = \begin{pmatrix} u & v & 0 & 0 & \cdots & 0 & 0 \\ 0 & u & v & 0 & \cdots & 0 & 0 \\ 0 & 0 & u & v & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & u & v \end{pmatrix}_{n\times(n+1)}.$$

Let $J = \cap_{n\geq 1} \operatorname{ann}_T M_n$. Then $\dim T/J = \dim T$.

Proof: Let $\hat{T}$ denote the m-adic completion of T. Then $\operatorname{ht}(u,v)\hat{T} = 2$, $\operatorname{ann}_T M_n = \operatorname{ann}_{\hat{T}}(M_n \otimes_T \hat{T}) \cap T$, and $\dim T/(I \cap T) \geq \dim \hat{T}/I$ for all ideals I of $\hat{T}$. Thus, we may assume T is complete. Now let p be a prime ideal of T such that $\dim T/p = \dim T$. Since T is catenary, $\operatorname{ht}(u,v)T/p = 2$. Assume the lemma is true for complete domains. Then $\cap_{n\geq 1} \operatorname{ann}_{T/p}(M_n \otimes_T T/p) = p/p$. Hence

$$\begin{aligned} J &= \cap_{n\geq 1} \operatorname{ann}_T M_n \\ &\subseteq \cap_{n\geq 1} \operatorname{ann}_T (M_n \otimes_T T/p) \\ &= p, \end{aligned}$$

which implies that $\dim T/J \geq \dim T/p = \dim T$. Thus, it suffices to prove the lemma for complete domains.

As T is complete, the integral closure S of T is a finite T-module ([Mat, page 263]). Since $\operatorname{ht}(u,v)S = 2$ ([Mat, Theorem 15.6]) and S is normal, $\{u, v\}$ is a regular sequence on S. It is easily seen that $I_n(A_n)$, the ideal of $n \times n$ minors of A_n, is $(u,v)^n T$. By the main result of [BE] we obtain $\operatorname{ann}_S(M_n \otimes_T S) = (u,v)^n S$. Hence $\operatorname{ann}_T M_n \subseteq (u,v)^n S \cap T$. As S is a finite T-module there exists an integer k such that $\operatorname{ann}_T M_n \subseteq (u,v)^{n-k} T$ for all $n \geq k$. Therefore, $\cap_{n\geq 1} \operatorname{ann}_T M_n = (0)$, which completes the proof. □

Lemma 2.8. *Assume $d \geq 2$ and let $f \in R$ be a homogeneous element of degree p such that the coefficients of f form a system of parameters for T. Then* $\dim T/\operatorname{ann}_T H^n_I(R/fR) \geq 2$.

Proof: Let $c_1, \ldots, c_d$ be the nonzero coefficients of f. Let $T' = T/(c_3, \ldots, c_d)T$ and $R' = T'[x_1, \ldots, x_n] \cong R/(c_3, \ldots, c_d)R \cong R \otimes_T T'$. Since

$$\begin{aligned} \dim T/\operatorname{ann}_T H^n_I(R/fR) &\geq \dim T/\operatorname{ann}_T(H^n_I(R/fR) \otimes_T T') \\ &= \dim T'/\operatorname{ann}_{T'} H^n_{IR'}(R'/fR'), \end{aligned}$$

we may assume that $\dim T = 2$ and f has exactly two nonzero terms.

For any $w \in R$ there is a surjective map $H^n_I(R/wfR) \to H^n_I(R/fR)$. Hence, $\operatorname{ann}_T H^n_I(R/wfR) \subseteq \operatorname{ann}_T H^n_I(R/fR)$. Thus, we may assume that the terms of f have no (nonunit) common factor. Without loss of generality, we may write $R = T[x_1, \ldots, x_k, y_1, \ldots y_r]$ and $f = ux_1^{d_1} \cdots x_k^{d_k} + vy_1^{e_1} \cdots y_r^{e_r} = u\mathbf{x}^{\mathbf{d}} + v\mathbf{y}^{\mathbf{e}}$, where $\{u, v\}$ is a system of parameters for T. As f is homogeneous, $p = \sum_i d_i = \sum_i e_i$.

Applying the right exact functor $H^n_I(\cdot)$ to $R(-p) \xrightarrow{f} R \to R/fR \to 0$ we obtain the exact sequence

$$H^n_I(R)_{-\ell-p} \xrightarrow{f} H^n_I(R)_{-\ell} \to H^n_I(R/fR)_{-\ell} \to 0$$

for each $\ell \in \mathbb{Z}$. For each ℓ, $H^n_I(R)_{-\ell}$ is a free T-module with basis

$$\{\mathbf{x}^{-\alpha}\mathbf{y}^{-\beta} \mid \sum_{i,j} \alpha_i + \beta_j = \ell, \alpha_i > 0, \beta_j > 0 \ \forall\ i,j\}$$

(e.g., [BS, Example 12.4.1]). Let q be an arbitrary positive integer and let $\ell(q) = qp + k + r$. Define $L_{-\ell(q)}$ to be the free T-summand of $H^n_I(R)_{-\ell(q)}$ spanned by the set

$$\{\mathbf{x}^{-s\mathbf{d}-\mathbf{1}}\mathbf{y}^{-t\mathbf{e}-\mathbf{1}} \mid s+t = q, s,t \geq 0\}.$$

Then the cokernel of $\delta_q : L_{-\ell(q+1)} \xrightarrow{f} L_{-\ell(q)}$ is a direct summand (as a T-module) of $H^n_I(R/fR)_{-\ell(q)}$. For a given q we order the basis elements for $L_{-\ell(q)}$ as follows:

$$x^{-s\mathbf{d}-\mathbf{1}}\mathbf{y}^{-t\mathbf{e}-\mathbf{1}} > x^{-s'\mathbf{d}-\mathbf{1}}\mathbf{y}^{-t'\mathbf{e}-\mathbf{1}}$$

if and only if $s > s'$. With respect to these ordered bases, the matrix representing δ_q is

$$\begin{pmatrix} u & v & 0 & 0 & \cdots & 0 & 0 \\ 0 & u & v & 0 & \cdots & 0 & 0 \\ 0 & 0 & u & v & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & u & v \end{pmatrix}_{(q+1)\times(q+2)}.$$

By Lemma 2.7, if $J = \cap_{q\geq 1} \operatorname{ann}_T \operatorname{coker} \delta_q$ then $\dim T/J = \dim T = 2$. As $\operatorname{coker} \delta_q$ is a direct T-summand of $H^n_I(R/fR)$, we have $\operatorname{ann}_T H^n_I(R/fR) \subseteq J$. This completes the proof. □

Lemma 2.9. *Under the assumptions of Lemma 2.8,* $\operatorname{Hom}_R(R/I, H^n_I(R/fR))$ *is not finitely generated as an* R*-module. Consequently,* $\operatorname{Hom}_R(R/I, H^n_I(R/fR))_k \neq 0$ *for infinitely many* k*.*

Proof: Suppose $\operatorname{Hom}_R(R/I, H^n_I(R/fR))$ is finitely generated. By Lemma 3.5 of [MV] we have that $I + \operatorname{ann}_R H^n_I(R/fR)$ is P-primary. (One should note that the hypothesis in [MV, Lemma 3.5] that the ring be complete is not necessary.) This implies that $\operatorname{ann}_R H^n_I(R/fR) \cap T = \operatorname{ann}_T H^n_I(R/fR)$ is m-primary, contradicting Lemma 2.8. □

We now give the proof of our main result:

Proof of Theorem 2.3: By Corollary 2.6, $\operatorname{Supp}_R H^n_I(R/fR) = \{P\}$. Thus, $\operatorname{Hom}_R(R/I, H^n_I(R/fR))_k$ has finite length as a T-module for all k and is nonzero for infinitely many k by Lemma 2.9. Consequently,

$$\operatorname{Hom}_R(R/P, H^n_I(R/fR))_k = \operatorname{Hom}_T(T/m, \operatorname{Hom}_R(R/I, H^n_I(R/fR))_k)$$

is nonzero for infinitely many k. Hence

$$^*\operatorname{soc}_R(H^n_I(R/fR)) = \operatorname{Hom}_R(R/P, H^n_I(R/fR))$$

is not finitely generated. □

References

[BKS] Brodmann, M., Katzman, M. and Sharp, R., *Associated primes of graded component of local cohomology modules*, Trans. Amer. Math. Soc., **354**, 4261–4283 (2002).

[BS] Brodmann, M. and Sharp, R., *Local Cohomology: an algebraic introduction with geometric applications*, Cambridge Studies in Advanced Mathematics no. **60**, Cambridge, Cambridge University Press, 1998.

[BH] Bruns, W. and Herzog, J., *Cohen-Macaulay Rings*, Cambridge Studies in Advanced Mathematics no. **39**, Cambridge, Cambridge University Press, 1993.

[BE] Buchsbaum, D. and Eisenbud, D., *What annihilates a module?*, J. Algebra **47**, 231–243 (1977).

[DM] Delfino, D. and Marley, T., *Cofinite modules and local cohomology*, J. Pure Appl. Algebra **121**, 45–52 (1997).

[Ha] Hartshorne, R., *Affine Duality and Cofiniteness*, Invent. Math. **9**, 145–164 (1970).

[HM] Helm, D. and Miller, E., *Bass numbers of semigroup-graded local cohomology*, Pacific J. Math. **209**, 41–66 (2003).

[Hu] Huneke, C., *Problems on local cohomology*, Free Resolutions in commutative algebra and algebraic geometry (Sundance, Utah, 1990), Research Notes in Mathematics **2**, Boston, MA, Jones and Bartlett Publishers, 1994, 993–108.

[HK] Huneke, C. and Koh, J., *Cofiniteness and vanishing of local cohomology modules*, Math. Proc. Cambridge Philos. Soc. **110**, 421–429 (1991).

[HS] Huneke, C. and Sharp, R., *Bass Numbers of local cohomology modules*, Trans. Amer. Math. Soc. **339**, 765–779 (1993).

[Ka] Katzman, M., *An example of an infinite set of associated primes of a local cohomology module*, J. Algebra, **252**, 161–166 (2002).

[KS] Katzman, M. and Sharp, R., *Some properties of top graded local cohomology modules*, J. Algebra **259**, 599–612 (2003).

[L1] Lyubeznik, G., *Finiteness Properties of local cohomology modules (An application of D-modules to commutative algebra)*, Invent. Math. **113**, 41–55 (1993).

[L2] Lyubeznik, G., *Finiteness properties of local cohomology modules for regular local rings of mixed characteristic: The unramified case*, Comm. Algebra **28** no. 12, 5867–5882 (2000).

[MV] Marley, T. and Vassilev, J., *Cofiniteness and associated primes of local cohomology modules*, J. Algebra **256**, 180–193 (2002).

[Mat] Matsumura, H., *Commutative Ring Theory*, Cambridge Studies in Advanced Mathematics no. **8**, Cambridge, Cambridge University Press, 1986.

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