Frobenius and modules of finite flat dimension

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Theorem (Peskine-Szpiro, 1973)

If M is a finitely generated R-module and $\operatorname{pd}_R M < \infty$ then $\operatorname{Tor}_i^R(R^{(e)}, M) = 0$ for all $i, e \geq 1$. In particular, $\operatorname{pd}_{R^{(e)}} R^{(e)} \otimes_R M < \infty$ for all $e \geq 1$.

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Tensoring $C(\mathbf{x}; R)$ with $R^{(e)}$ gives the Čech complex $C(\mathbf{x}^{p^e}, R^{(e)})$, which is acyclic as $\mathbf{x}^{p^e}$ is a regular sequence on $R^{(e)}$.

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In particular, $\operatorname{Tor}_i^R(R^{(e)}, H_I^r(R)) = 0$ for all $i, e \geq 1$.

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Theorem (Herzog, 1974)

Suppose R has finite Krull dimension and M a finitely generated R-module. If $\mathrm{Tor}_i^R(R^{(e)}, M) = 0$ for all $i > 0$ and infinitely many e. Then $\mathrm{pd}_R\, M < \infty$.

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Question

Does Herzog's theorem hold for arbitrary modules?

Answers!

Main Theorem

Let M be an R-module.

(a) If $\operatorname{fd}_R M < \infty$ then $\operatorname{Tor}_i^R(R^{(e)}, M) = 0$ for all $i, e > 0$ and $\operatorname{fd}_R M = \operatorname{fd}_{R^{(e)}} R^{(e)} \otimes_R M$ for all $e > 0$.

(b) If R has finite Krull dimension and $\operatorname{Tor}_i^R(R^{(e)}, M) = 0$ for all $i > 0$ and infinitely many e, then $\operatorname{pd}_R M < \infty$.

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(b) If R has finite Krull dimension and $\operatorname{Tor}_i^R(R^{(e)}, M) = 0$ for all $i > 0$ and infinitely many e, then $\operatorname{pd}_R M < \infty$.

If in addition $R^{(1)}$ is finitely generated as an R-module, we have:

(c) If $\operatorname{id}_R M < \infty$ then $\operatorname{Ext}_R^i(R^{(e)}, M) = 0$ for all $i, e > 0$ and $\operatorname{id}_R M = \operatorname{id}_{R^{(e)}} \operatorname{Hom}_R(R^{(e)}, M)$ for all $e > 0$.

(d) If R has finite Krull dimension and $\operatorname{Ext}_R^i(R^{(e)}, M) = 0$ for all $i > 0$ and infinitely many e, then $\operatorname{id}_R M < \infty$.

Flat covers

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Theorem (J. Xu, 1995; Bican, El-Bashir, Enochs, 2001)

Flat covers exist!

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Facts:

- The kernel of a flat cover is cotorsion.
- A flat cover of a cotorsion module is cotorsion.
- If R is complete then any Artinian or Noetherian R-module is cotorsion.

Decomposition of flat cotorsion modules

Theorem (Enochs, 1984)

Let F be a flat cotorsion R-module. Then

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$$F \cong \prod_{q \in \operatorname{Spec} R} T(q),$$

where $T(q)$ is the completion with respect to the qR_q-adic topology of a free R_q-module $G(q)$.

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For $q \in \operatorname{Spec} R$ we denote of the rank of $G(q)$ by $\pi(q, F)$.

Minimal flat resolutions

Definition (Enochs-Xu, 1997)

A *minimal flat resolution* of M is an acyclic complex $(\mathbf{F}, \partial)$ such that each F_i is flat, $F_i = 0$ for all $i < 0$, $H_0(\mathbf{F}) \cong M$, and $F_i \to \operatorname{coker} \partial_{i+1}$ is a flat cover for all i.

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For $q \in \operatorname{Spec} R$ and $i \geq 1$ we let $\pi_i(q, M) := \pi(q, F_i)$. We set $\pi_0(q, M) := \pi(q, C_R(F_0))$, where $C_R(F_0)$ is the cotorsion envelope of F_0. We call the $\pi_i(q, M)$ the *Enochs-Xu numbers* of M.

Enochs-Xu Theorem

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Let M be an R-module.

1. $\pi_i(q, M) = \pi_i(q, C_R(M))$ for all i, $q \in \operatorname{Spec} R$.

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3. If M is cotorsion then for all $i \geq 0$ and $q \in \operatorname{Spec} R$, $\pi_i(q, M) = \dim_{k(q)} \operatorname{Tor}_i^{R_q}(k(q), \operatorname{Hom}_R(R_q, M))$.

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Note: Part (1) implies that $\operatorname{fd}_R M = \operatorname{fd}_R C_R(M)$.

Peskine and Szpiro's Acyclicity Lemma

Definition

Let (R, m) be local and M an R-module. We define the *depth* of M by

$$\operatorname{depth} M := \inf\{i \geq 0 \mid H^i_m(M) \neq 0\}.$$

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Lemma (Peskine-Szpiro, 1973)

Let R be a local ring and consider a bounded complex $\mathbf{T}$ of R-modules: $0 \to T_s \to T_{s-1} \to \cdots \to T_0 \to 0$. Suppose the following two conditions hold for each $i > 0$:

1. $\operatorname{depth} T_i \geq i$;
2. $\operatorname{depth} \mathrm{H}_i(\mathbf{T}) = 0$ or $\mathrm{H}_i(\mathbf{T}) = 0$.

Then $\mathrm{H}_i(\mathbf{T}) = 0$ for all $i > 0$.

Key result: A vanishing result for Enochs-Xu numbers

Theorem

Suppose $\operatorname{fd}_R M < \infty$. Then $\pi_i(p, M) = 0$ for all $p \in \operatorname{Spec} R$ and $i > \operatorname{depth} R_p$.

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Corollary

Suppose (R, m) is local and $\operatorname{fd}_R M < \infty$. Suppose $\mathbf{F}$ is a minimal flat resolution of M. Then $\operatorname{depth}_{R^{(e)}} R^{(e)} \otimes_R F_i = \infty$ for $i > \operatorname{depth} R$.

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Proof.

Suppose $i > \operatorname{depth} R$. Then $H^j_m(R^{(e)} \otimes_R F_i) \cong H^j_m(R^{(e)}) \otimes_R F_i = 0$ for all j, since F_i is flat and a product of R_p-modules, $p \neq m$. □

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Suppose $\operatorname{fd}_R M < \infty$ but $\operatorname{Tor}_i^R(R^{(e)}, M) \neq 0$ for some $i, e > 0$.

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Let $\mathbf{F}$ be a minimal flat resolution of M and $\mathbf{G} = R^{(e)} \otimes_R \mathbf{F}$.
Then for $i > 0$, $\operatorname{depth} \mathrm{H}_i(\mathbf{G}) = 0$ or $\mathrm{H}_i(\mathbf{G}) = 0$.

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If $i > \operatorname{depth} R$ then $\operatorname{depth} G_i = \infty$ by the Corollary.
Hence, $\mathbf{G} = R^{(e)} \otimes_R \mathbf{F}$ is acyclic by Peskine and Szpiro's acyclicity lemma. Thus, $\text{Tor}_i^R(R^{(e)}, M) = 0$ for $i > 0$, a contradiction.

Thank you!