The Frobenius Functor and Injective Modules

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$$r \cdot s := rs \qquad \text{(left } R\text{-action)}$$
$$s \cdot r := sr^p \qquad \text{(right } R\text{-action)}$$

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for $r \in R$ and $s \in R^f$.

Then $F_R(-) := R^f \otimes_R -$ is a functor from the category of left R-modules to itself.

F_R is called the *Frobenius functor* of R.

Properties of F_R

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Since

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we have $F_R(\phi) : R^n \xrightarrow{A^{[p]}} R^m$ where $A^{[p]} = [a_{ij}^p]$.

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$$R^n \xrightarrow{[a_1^p \ldots a_n^p]} R \to F_R(R/I) \to 0$$

to conclude that $F_R(R/I) \cong R/I^{[p]}$ where $I^{[p]} = (a_1^p, \ldots, a_n^p)$.

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The Frobenius functor has been used with great success to deepen our understanding of local rings of characteristic p:

- The homological conjectures
- Tight closure
- Uniform bounds
- Symbolic powers
- Local cohomology

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Definition

A ring R is said to be *FPI* if $F_R(I)$ is injective for every injective R-module I. We say R is *weakly FPI* if $F_R(I)$ is injective for every Artinian injective R-module I.

Matlis decomposition

By Matlis's decomposition theorem for injective modules, if I is injective then

$$I \cong \bigoplus_{P \in \operatorname{Spec} R} E_R(R/P)^{\alpha_P}.$$

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- R is weakly FPI if and only if $F_R(E_R(R/m))$ is injective for every maximal ideal m of R.

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- R is weakly FPI if and only if $F_R(E_R(R/m))$ is injective for every maximal ideal m of R.
- R is FPI if and only if R_P is weakly FPI for every $P \in \operatorname{Spec} R$.

Gorenstein rings are FPI

Remark

Let M be an R-module of dimension s and I an ideal of R. Then $F_R(H_I^s(M)) \cong H_I^s(F_R(M))$.

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Proof: Since $H_I^i(M) = 0$ for any $i > s$, one has

$$R^f \otimes_R H_I^s(M) \cong H_{I^{[p]}}^s(R^f \otimes_R M) \cong H_I^s(F_R(M)).$$

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Theorem (Huneke-Sharp, 1993)

If R is Gorenstein then R is FPI.

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Theorem (Huneke-Sharp, 1993)

If R is Gorenstein then R is FPI.

Proof: Since Gorenstein localizes, it suffices to show R is weakly FPI when (R, m) is a local Gorenstein ring. In this case, $E_R(R/m) \cong H_m^d(R)$, $d = \dim R$. Apply the remark with $M = R$.

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Answer: No. In fact, every quasi-Gorenstein ring is FPI, and there exist quasi-Gorenstein rings which are not Cohen-Macaulay. (A local ring R is *quasi-Gorenstein* if $H^d_m(R) \cong E_R(R/m)$.)

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Proposition

Every zero-dimensional FPI ring is Gorenstein.

Proof: We may assume (R, m) is local. Let $E = E_R(R/m)$ and consider a presentation $R^n \xrightarrow{A} R^\ell \to E \to 0$. Repeated applications of F_R gives the exact sequence: $R^n \xrightarrow{A^{[q]}} R^\ell \to F^e_R(E) \to 0$, where $q = p^e$. Since R is zero-dimensional, $A^{[q]} = 0$ for some q. Hence, $R^\ell \cong F^e_R(E)$, which is injective as R is FPI. Hence R is injective.

Properties of FPI rings

Theorem A

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Theorem B

If R is weakly FPI then R satisfies Serre's condition S_1.

Theorem C

Let (R, m) be a local ring and $E := E_R(R/m)$. TFAE:

1. R is weakly FPI and $\operatorname{Tor}_i^R(R^f, E) = 0$ for $1 \leq i \leq \operatorname{depth} R$;
2. R is Gorenstein.

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If R is local then R is weakly FPI if and only if $\hat{R}$ is weakly FPI.

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Corollary

If R is local and a quotient of a Gorenstein ring then R is FPI if and only if $\hat{R}$ is FPI.

One-dimensional FPI rings

Theorem E

Let R be a one-dimensional local ring. TFAE:

1. R is weakly FPI;
2. R is FPI;
3. R is Cohen-Macaulay and has a canonical ideal ω_R such that $\omega_R \cong \omega_R^{[p]}$.

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Example

Let $R = \mathbb{F}_p[x, y, z]/(xy, xz, yz)$. Then R is a one-dimensional FPI ring which is not Gorenstein.

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Example

Let $R = \mathbb{F}_p[x, y, z]/(xy, xz, yz)$. Then R is a one-dimensional FPI ring which is not Gorenstein.

Proof: One can show that $\omega_R = (y - x, z - x)$ is a canonical ideal. Also, $\omega_R^{[p]} = (x + y + z)^{p-1}\omega_R \cong \omega_R$.

One-dimensional FPI rings

Theorem F

Suppose R is a one-dimensional complete local ring with algebraically closed residue field. Assume R has at most two associated primes. Then R is (weakly) FPI if and only if R is Gorenstein.

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Example

Let $R = \overline{\mathbb{F}_p}[[t^3, t^4, t^5]]$. Then R is not weakly FPI.

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Proof: Let $E = E_R(R/m)$. One can show by direct computation that the socle dimension of $F_R(E)$ is 3. Hence, $F_R(E) \ncong E$.

Proof of Theorem B

We can assume (R, m) is local and complete. Let $E := E_R(R/m)$ and for $P \in \operatorname{Spec} R$ set $E_{R/P} := \operatorname{Hom}_R(R/P, E) = E_{R/P}(R/m)$. Using Matlis duality and facts about local cohomology, one can show that for all $q = p^e$

$$\operatorname{Ann}_R F_R^e(E_{R/P}) \subseteq P^{[q]} R_P \cap R.$$

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Now let $P \in \operatorname{Ass}_R R$. Then there is an exact sequence $0 \to R/P \to R$. Applying $\operatorname{Hom}_R(-, E)$, we have an exact sequence $E \to E_{R/P} \to 0$, where $E = E_R(R/m)$.

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Now let $P \in \operatorname{Ass}_R R$. Then there is an exact sequence $0 \to R/P \to R$. Applying $\operatorname{Hom}_R(-, E)$, we have an exact sequence $E \to E_{R/P} \to 0$, where $E = E_R(R/m)$. Applying Frobenius e times, we have

$$E \cong F_R^e(E) \to F_R^e(R/P) \to 0.$$

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Thank you!