Frobenius and homological dimensions

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$$\begin{aligned} r \cdot s &:= f^e(r)s \\ &= r^{p^e}s \in {}^eR \end{aligned}$$

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Corollary (Kunz, 1969)

If R is regular then ${}^eR$ is a flat R-module for all $e > 0$.

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- R is Cohen-Macaulay of positive dimension and $\mathrm{Tor}_i^R({}^eR, M) = 0$ for dim R consecutive $i > 0$ for some e sufficiently large. (Miller, 2001)
- R is a complete intersection and $\mathrm{Tor}_i^R({}^eR, M) = 0$ for some $i > 0$ and some $e > 0$. (Avramov-Miller, 2001).

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- Let J be a minimal injective resolution of M. Let $S = {}^eR$ and $\mathfrak{n}$ the maximal ideal of S. Then

$$\mathrm{Hom}_R(S, J^0) \to \mathrm{Hom}_R(S, J^1) \to \cdots \to \mathrm{Hom}_R(S, J^{d+1}) \to G$$

is the start of an injective resolution of $\mathrm{Hom}_R(S, M)$.

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By minimality, we have $\phi = 0$. Hence, τ is injective.

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Using induction, one can assume $\operatorname{Supp}_R J^{d+1} \subseteq \{\mathfrak{m}\}$. Thus, $\operatorname{Supp}_S \operatorname{Hom}_R(S, J^{d+1}) \subseteq \{\mathfrak{n}\}$.

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Lemma: If $f : I \to I'$ is a map of injective R-modules and $f_* : \operatorname{Hom}_R(S, I) \to \operatorname{Hom}_R(S, I')$ is surjective, where S is a f.g. faithful R-module, then f is surjective.

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Hence, $J^{d-1} \to J^d$ is surjective and $\operatorname{id}_R M < \infty$.

The End

Thank you!