# Characterizing Gorenstein rings using the Frobenius endomorphism

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# Frobenius and regularity

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- $R$ is regular if and only if $\operatorname{fd}_R {}^eM < \infty$ for some $e > 0$ and some nonzero f.g. $R$-module $M$. (Avramov-Hochster-Iyengar-Yao, 2012)

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In terms of vanishing of Tor, we have:

- $R$ is regular if and only if $\operatorname{Tor}_1^R(k, {}^eR) = 0$ for some $e > 0$.
- $R$ is regular if and only if $\operatorname{Tor}_i^R(k, {}^eR) = 0$ for some $i, e > 0$.
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If $R$ is a complete intersection and $M$ a f.g. $R$-module, then $\operatorname{pd}_R M < \infty$ if and only if $\operatorname{Tor}_i^R(M, {}^eR) = 0$ for some $i, e > 0$. (Avramov-Miller, 2001)

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However, this does not hold in general, even for Gorenstein rings. (Dao-Li-Miller, 2010)

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- $R$ is Gorenstein if and only if $\mathrm{Ext}^i_R({}^eR, R) = 0$ for some $e$ large enough and all $i$ sufficiently large. (Takahashi-Yoshino, 2004)

## Frobenius and Gorenstein

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**Question**

Can we find characterizations of Gorenstein in terms of a finite number of vanishings of $\operatorname{Tor}_i^R(M, {}^eR)$ or $\operatorname{Ext}^i_R({}^eR, M)$ for some $R$-module or complex $M$?

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The Homotopical Loewy length of an $R$-complex $M$ is the number

$$\ell\ell_{\mathsf{D}(R)}(M) := \inf\{\ell\ell_R(V) \mid M \simeq V \text{ in } \mathsf{D}(R)\}.$$

# Homotopical Loewy length

Let $\boldsymbol{x}$ be a system of parameters for $R$ and $M$ an $R$-complex $M$. We write $\mathrm{K}[\boldsymbol{x}; M]$ for the Koszul complex on $\boldsymbol{x}$, with coefficients in $M$.

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In [AHIY, 2012] it is proved that

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We define

$$c(R) := \inf\{\ell\ell_{\mathsf{D}(R)} \mathsf{K}[\boldsymbol{x}; R] \mid \boldsymbol{x} \text{ is an s.o.p. for } R\}.$$

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If $R$ is Cohen-Macaulay then $c(R_{\mathfrak{p}}) \leq e(R)$ for all $\mathfrak{p} \in \operatorname{Spec} R$, where $e(R)$ is the multiplicity of $R$.

# finite flat dimension

**Theorem (Dailey-Iyengar-M, 2017)**

Let $M$ be an $R$-complex and $d = \dim R$. Suppose $s := \sup M < \infty$. TFAE:

1. $\mathrm{fd}_R\, M < \infty$.
2. $\mathrm{Tor}_i^R(M, {}^eR) = 0$ for all $i > s, e > 0$.
3. There exists $t \geq s$ such that $\mathrm{Tor}_i^R(M, {}^eR) = 0$ for $t \leq i \leq t + d$ for infinitely many $e$.

**Remarks:**

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- If $\operatorname{supp}_R M = \{\mathfrak{p} \in \operatorname{Spec} R \mid k(\mathfrak{p}) \otimes^{\mathbf{L}}_R M \not\simeq 0\} = \{\mathfrak{m}\}$ then vanishing for a single $e > \log_p c(R)$ is sufficient.

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- If $H_i(M)$ is finitely generated for all $i$ then vanishing for a single $e > \log_p c(R)$ is sufficient.
- If $\operatorname{supp}_R M = \{\mathfrak{p} \in \operatorname{Spec} R \mid k(\mathfrak{p}) \otimes^{\mathbf{L}}_R M \not\simeq 0\} = \{\mathfrak{m}\}$ then vanishing for a single $e > \log_p c(R)$ is sufficient.
- If $R$ is Cohen-Macaulay, then vanishing for a single $e > \log_p e(R)$ is sufficient.

# finite flat dimension, part II

**Theorem (Falahola - M)**

Let $M$ be an $R$-complex and $d = \dim R$. Suppose $s = \sup M < \infty$, and assume either $H_i(M)$ is f.g. for each $i$ or $\operatorname{supp}_R M = \{\mathfrak{m}\}$. TFAE:

1. $\operatorname{fd}_R M < \infty$.
2. $\operatorname{Ext}^i_R(M, {}^eR) = 0$ for all $i > s, e > 0$.
3. There exists $t \geq s$ such that $\operatorname{Ext}^i_R(M, {}^eR) = 0$ for $t \leq i \leq t + d$ for some $e > \log_p c(R)$.

# finite injective dimension

**Theorem (Falahola - M)**

Let $M$ be an $R$-complex and $d = \dim R$. Suppose $M$ is a bounded complex and that one of the following holds: (a) $H(M)$ is f.g.; (b) $\operatorname{supp}_R M = \{m\}$; or (c) $R$ is $F$-finite. Let $\ell = -\inf M$.

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1. $\operatorname{id}_R M < \infty$
2. $\operatorname{Ext}^i_R({}^eR, M) = 0$ for all $i > \ell, e > 0$.
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3. There exists a $t \geq \ell$ such $\operatorname{Ext}^i_R({}^eR, M) = 0$ for $t \leq i \leq t + d$ for infinitely many $e$.

Moreover, a single large $e$ is sufficient in condition (3) in cases (a), (b) or if $R$ is CM.

# Some Gorenstein characterizations

**Theorem (Foxby, 1979)**

Let $(R, \mathfrak{m})$ be a local ring. TFAE:

1. $R$ is Gorenstein.
2. There exists a complex $M$ with $m \in \operatorname{supp}_R M$ such that $\operatorname{fd}_R M < \infty$ and $\operatorname{id}_R M < \infty$.

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3. $\operatorname{Ext}_R^i({}^eR, R) = 0$ for $d+1$ consecutive positive values of $i$ and some $e > \log_p c(R)$.
4. $R$ is CM with a canonical module $\omega_R$ and $\operatorname{Tor}_i^R(\omega_R, {}^eR) = 0$ for $d+1$ consecutive positive values of $i$ and some $e > \log_p c(R)$.

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6. $\mathrm{Ext}_R^i({}^eR, C) = 0$ for some $d+1$ consecutive values of $i > d$ and $e > \log_p c(R)$, where $C$ is the Cech complex on an s.o.p.

# The End

Thank you!