COFINITE MODULES AND LOCAL COHOMOLOGY

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Abstract. We show that if M is a finitely generated module over a commutative Noetherian local ring R and I is a dimension one ideal of R (i.e., $\dim R/I = 1$), then the local cohomology modules $H^i_I(M)$ are I-cofinite; that is, $\operatorname{Ext}^j_R(R/I, H^i_I(M))$ is finitely generated for all i, j. We also show that if R is a complete local ring and P is a dimension one prime ideal of R, then the set of P-cofinite modules form an abelian subcategory of the category of all R-modules. Finally, we prove that if M is an n-dimensional finitely generated module over a Noetherian local ring R and I is any ideal of R, then $H^n_I(M)$ is I-cofinite.

Let R be a commutative Noetherian local ring with maximal ideal m and let I be an ideal of R. An R-module N is said to be *I-cofinite* if $\operatorname{Supp} N \subseteq \operatorname{V}(I)$ and $\operatorname{Ext}^i_R(R/I, N)$ is finitely generated for all $i \geq 0$. Using Matlis duality one can show that a module is m-cofinite if and only if it is Artinian. As a consequence, the local cohomology modules $H^i_m(M)$ are m-cofinite for any finitely generated R-module M. In [6], Hartshorne posed the question of whether this statement still holds when m is replaced by an arbitrary ideal I; i.e., is $H^i_I(M)$ I-cofinite for all i? In general, the answer is no, even if R is a regular local ring. Let $R = k[[x, y, u, v]]$ be the formal power series ring in four variables over a field k, m the maximal ideal of R, $P = (x, u)R$ and $M = R/(xy - uv)$. Hartshorne showed that $\operatorname{Hom}_R(R/m, H^2_P(M))$ is not finitely generated, and hence $\operatorname{Hom}_R(R/P, H^2_P(M))$ cannot be finitely generated. In the positive direction, Hartshorne proved that if R is a complete regular local ring, P a dimension one prime ideal of R, and M a finitely generated R-module, then $H^i_P(M)$ is P-cofinite for all i. In 1991, Huneke and Koh proved that if R is a complete local Gorenstein domain, I a dimension one ideal of R, and M a finitely generated R-module, then $H^i_I(M)$ is I-cofinite for all i ([7, Theorem 4.1]. Recently, Delfino proved that the Gorenstein hypothesis in the Huneke-Koh theorem may be weakened to include all complete local domains R which satisfy one of the following conditions: (1) R contains a field; (2) if q is a uniformizing parameter for a coefficient ring for R then either $q \in \sqrt{I}$ or q is not in any prime minimal over I; or (3) R is Cohen-Macaulay ([3, Theorem 3] and [4, Theorem 2.21]). In this paper, we eliminate the complete domain hypothesis entirely by proving the following:

Theorem 1. *Let R be a Noetherian local ring, I a dimension one ideal of R, and M a finitely generated R-module. Then $H^i_I(M)$ is I-cofinite for all i.*

We prove this by establishing a change of ring principle for cofiniteness (Proposition 2) and then applying it to the Huneke-Koh result. Using this change of ring principle,

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we are also able to generalize Hartshorne's theorem that over a regular local ring, the P-cofinite modules (P a dimension one prime) form an abelian subcategory of the category of R-modules (Theorem 2).

We also prove a cofiniteness result about $H^n_I(M)$, where M is a finitely generated R-module and $n = \dim M$. In [12], Sharp proved that if R is a Noetherian local ring of dimension d and I is any ideal of R, then $H^d_I(R)$ is Artinian. From this it follows easily that if M is a finitely generated R-module of dimension n then $H^n_I(M)$ is Artinian. (See also [10, Theorem 2.2].) Thus, $H^n_I(M)$ is m-cofinite. We prove that $H^n_I(M)$ is in fact I-cofinite (Theorem 3).

We begin the proof of Theorem 1 by proving the following generalization of [7, Lemma 4.2] and [3, Lemma 2]:

Proposition 1. *Let R be a Noetherian ring, M a finitely generated R-module and N an arbitrary R-module. Suppose that for some $p \geq 0$, $\operatorname{Ext}^i_R(M,N)$ is finitely generated for all $i \leq p$. Then for any finitely generated R-module L with $\operatorname{Supp} L \subseteq \operatorname{Supp} M$, $\operatorname{Ext}^i_R(L,N)$ is finitely generated for all $i \leq p$.*

proof. Using induction on p, we may assume that $\operatorname{Ext}^i_R(L,N)$ is finitely generated for all $i < p$ and all finitely generated modules L with $\operatorname{Supp} L \subseteq \operatorname{Supp} M$. (This is satisfied vacuously if $p = 0$.) By Gruson's Theorem ([13, Theorem 4.1]), given any finitely generated R-module L with $\operatorname{Supp} L \subseteq \operatorname{Supp} M$ there exists a finite filtration

$$0 = L_0 \subset L_1 \subset \cdots \subset L_n = L$$

such that the factors L_i/L_{i-1} are homomorphic images of a direct sum of finitely many copies of M. By using short exact sequences and induction on n, it suffices to prove the case when $n = 1$. Thus, we have an exact sequence of the form

$$0 \longrightarrow K \longrightarrow M^n \longrightarrow L \longrightarrow 0$$

for some positive integer n and some finitely generated module K. This gives the long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^{p-1}_R(K,N) \longrightarrow \operatorname{Ext}^p_R(L,N) \longrightarrow \operatorname{Ext}^p_R(M^n,N) \longrightarrow \cdots$$

Since $\operatorname{Supp} K \subseteq \operatorname{Supp} M$ we have that $\operatorname{Ext}^{p-1}_R(K,N)$ is finitely generated (by the induction on p). As $\operatorname{Ext}^p_R(M^n,N) \cong \operatorname{Ext}^p_R(M,N)^n$ is finitely generated, the result follows.

As a consequence, we have the following:

Corollary 1. *Let R be a Noetherian ring, I an ideal of R and N an R-module. The following are equivalent:*

(a) $\operatorname{Ext}^i_R(R/I,N)$ *is finitely generated for all* $i \geq 0$,
(b) $\operatorname{Ext}^i_R(R/J,N)$ *is finitely generated for all* $i \geq 0$ *and ideals* $J \supseteq I$,
(c) $\operatorname{Ext}^i_R(R/P,N)$ *is finitely generated for all* $i \geq 0$ *and all primes* P *minimal over* I.

proof. We show that (c) implies (a). Let $P_1, \dots, P_n$ be the minimal primes of I and $M = R/P_1 \oplus \cdots \oplus R/P_n$. Then $\operatorname{Ext}^i_R(M,N)$ is finitely generated for all i. As $\operatorname{Supp} R/I = \operatorname{Supp} M$, $\operatorname{Ext}^i_R(R/I,N)$ is finitely generated for all i by the proposition.

The next result concerns spectral sequences, for which we use the notation from chapter 5 of [14]. The essential idea for this lemma can be found in the proof of [3, Theorem 3].

Lemma 1. *Let R be a Noetherian ring and $\{E_r^{pq}\}$ a first quadrant cohomology spectral sequence (starting with E_a, for some $a \geq 1$) converging to H^* in the category of R-modules. For a fixed integer n, suppose H^n is finitely generated and $E_a^{p,q}$ is finitely generated for all $p < n$ and $q \geq 0$. Then $E_a^{n,0}$ is finitely generated.*

proof. If $n = 0$ then $E_a^{00} = H^0$ is finitely generated. Suppose $n > 0$. First note that E_r^{pq} is finitely generated for any $p < n$, $q \geq 0$, and $r \geq 1$, since E_r^{pq} is a subquotient of E_a^{pq}. Also, as $E_\infty^{n,0}$ is a isomorphic to a submodule of H^n, $E_\infty^{n,0}$ is finitely generated. Now since $\{E_r^{pq}\}$ is a first quadrant spectral sequence (in particular, since there are no nonzero terms below the p-axis), there is an exact sequence

$$E_r^{n-r,r-1} \longrightarrow E_r^{n,0} \longrightarrow E_{r+1}^{n,0} \longrightarrow 0$$

for all $r \geq a$. As $E_r^{n,0} = E_\infty^{n,0}$ for sufficiently large r (and thus is finitely generated), we can work backwards to see that $E_r^{n,0}$ is finitely generated for all $r \geq a$.

We now prove the change of ring principle for cofiniteness:

Proposition 2. *Let R be a Noetherian ring and S a module finite R-algebra. Let I be an ideal of R and M an S-module. Then M is I-cofinite (as an R-module) if and only if M is IS-cofinite (as an S-module).*

proof. First note that $\operatorname{Supp}_R M \subseteq \mathrm{V}(I)$ if and only if $\operatorname{Supp}_S M \subseteq \mathrm{V}(IS)$. Now consider the Grothendieck spectral sequence (see [11, Theorem 11.65], for example)

$$E_2^{pq} = \operatorname{Ext}_S^p(\operatorname{Tor}_q^R(S, R/I), M) \Longrightarrow \operatorname{Ext}_R^{p+q}(R/I, M).$$

Suppose first that M is IS-cofinite. Then $E_2^{p,0} = \operatorname{Ext}_S^p(S/IS, M)$ is finitely generated for all p. Since $\operatorname{Supp}\operatorname{Tor}_q^R(S, R/I) \subseteq \operatorname{Supp} S/IS$ for all q, E_2^{pq} is finitely generated for all p and q by Proposition 1. Since the spectral sequence is bounded, it follows that $\operatorname{Ext}_R^n(R/I, M)$ is finitely generated for all n.

Conversely, suppose that M is I-cofinite. We use induction on n to show $E_2^{n,0} = \operatorname{Ext}_S^n(S/IS, M)$ is finitely generated. Now $E_2^{00} = \operatorname{Hom}_S(S/IS, M) \cong \operatorname{Hom}_R(R/I, M)$ is finitely generated. Suppose that $n > 0$ and $E_2^{p,0}$ is finitely generated for all $p < n$. By Proposition 1, E_2^{pq} is finitely generated for all $p < n$ and $q \geq 0$. Since $H^n = \operatorname{Ext}_R^n(R/I, M)$ is finitely generated, $E_2^{n,0}$ is finitely generated by Lemma 1.

As a final preparation for the proof of Theorem 1, we need the following fact:

Lemma 2. *Let (R, m) be a local ring and S the m-adic completion of R. Let I be an ideal of R and M an R-module. Then $H_I^i(M)$ is I-cofinite if and only if $H_{IS}^i(M \otimes_R S)$ is IS-cofinite.*

proof. Since $\operatorname{Ext}_R^i(R/I, H_I^i(M)) \otimes_R S \cong \operatorname{Ext}_S^i(S/IS, H_{IS}^i(M \otimes_R S)$, it is enough to see that an R-module N is finitely generated if and only if $N \otimes_R S$ is finitely generated as an S-modules. If N is finitely generated, the implication is obvious. If $N \otimes_R S$ is finitely generated then, using the faithful flatness of S, one can see that any ascending chain of submodules of N must stabilize.

Theorem 1 now follows readily:

proof of Theorem 1. By Lemma 2 we may assume R is complete. Thus, R is the homomorphic image of a regular local ring T. Let J be a dimension one ideal of T such that $JR = I$. Then $H^j_J(M)$ is J-cofinite by [7, Theorem 4.1] for all j. By Proposition 2, $H^j_I(M) \cong H^j_J(M)$ is I-cofinite for all j.

If N is an R-module then the ith Bass number of N with respect to p is defined by $\mu_i(p, N) = \dim_{k(p)} \operatorname{Ext}^i_{R_p}(k(p), N_p)$, where $k(p) = (R/p)_p$. If M is finitely generated and I is a zero-dimensional ideal then the Bass numbers of $H^i_I(M)$ are finite since $H^i_I(M)$ is Artinian. However, as Hartshorne's example shows, this does not hold for arbitrary ideals and modules, even over a complete regular local ring. In the special case that $M = R$, Huneke and Sharp proved that if R is a regular local ring of characteristic p and I is an ideal of R, then the Bass numbers of $H^i_I(R)$ are finite for all i ([8, Theorem 2.1]). Lyubeznik proved this same result in the case R is a regular local ring containing a field of characteristic 0 ([9, Corollary 3.6]). In [1], it is proved that if R is a complete local Gorenstein domain, I is a dimension one ideal and M is a Matlis reflexive R-module (i.e., $\operatorname{Hom}_R(\operatorname{Hom}_R(M, E), E) \cong M$ where $E = E_R(R/m)$), then the Bass numbers of $H^i_I(M)$ are finite. Using Theorem 1, we can prove the following:

Corollary 2. *Let R be a Noetherian ring, I a dimension one ideal of R, and M a finitely generated R-module. Then $\mu_i(p, H^j_I(M)) < \infty$ for all integers i, j and $p \in \operatorname{Spec}(R)$.*

proof. If $p \not\supseteq I$, then $\mu_i(p, H^j_I(M)) = 0$. If $p \supseteq I$ we can localize and assume $p = m$. By Theorem 1, $\operatorname{Ext}^i_R(R/I, H^j_I(M)$ is finitely generated for all i, j. Thus, $\operatorname{Ext}^i_R(R/m, H^j_I(M))$ is finitely generated for all i, j by Corollary 1.

Another question Hartshorne addressed in [6] was the following: if R is a complete regular local ring and P is a prime ideal, do the P-cofinite modules form an abelian subcategory of the category of all R-modules? That is, if $f\colon A \longrightarrow B$ is an R-module map of P-cofinite modules, are $\ker f$ and $\operatorname{coker} f$ P-cofinite? Hartshorne gave the following counterexample: let $R = k[[x, y, u, v]]$, $P = (x, u)$ and $M = R/(xy - uv)$ as in the example mentioned above. Applying the functor $H^0_P(-)$ to the exact sequence

$$0 \longrightarrow R \xrightarrow{xy-uv} R \longrightarrow M \longrightarrow 0$$

we get the exact sequence

$$\cdots \longrightarrow H^2_P(R) \xrightarrow{f} H^2_P(R) \longrightarrow H^2_P(M) \longrightarrow 0.$$

Since $H^j_P(R) = 0$ for all $j \neq 2$, one can show (using a collapsing spectral sequence) that $\operatorname{Ext}^i_R(R/P, H^2_P(R)) \cong \operatorname{Ext}^{i+2}_R(R/P, R)$ for all i. Thus, $H^2_P(R)$ is P-cofinite. However, as mentioned above, $\operatorname{coker} f = H^2_P(M)$ is not P-cofinite. On the positive side, Hartshorne proved that if P is a dimension one prime ideal of a complete regular local ring then the answer to his question is yes. Using Proposition 2, we can extend this result to arbitrary complete local rings:

Theorem 2. *Let R be a complete local ring and P a dimension one prime ideal of R. Then the P-cofinite modules form an abelian subcategory of the category of all R-modules.*

proof. Let $f\colon M \longrightarrow N$ be a map of P-cofinite modules. Since R is complete there exists a regular local ring T and a dimension one prime ideal Q of T such that R is a quotient of T and $QT = P$. Since M and N are Q-cofinite T-modules, $\ker f$ and $\operatorname{coker} f$ are Q-cofinite by Hartshorne's theorem ([6, Proposition 7.6]). Therefore, $\ker f$ and $\operatorname{coker} f$ are P-cofinite by Proposition 2.

We strongly believe Theorem 2 holds for dimension one ideals of an arbitrary local ring.

We now turn our attention to proving Theorem 3. The techniques are essentially those of Sharp ([12]) and Yassemi ([15]). Let (R, m) be a local ring, M an R-module, and $E = E_R(R/m)$ the injective hull of R/m. Following [15], we define a prime p to be a *coassociated prime* of M if p is an associated prime of $M^{\mathrm{v}} = \operatorname{Hom}_R(M, E)$. We denote the set of coassociated primes of M by $\operatorname{Coass}_R M$ (or simply $\operatorname{Coass} M$ if there is no ambiguity about the underlying ring). Note that $\operatorname{Coass} M = \emptyset$ if and only if $M = 0$. We first make a couple of preliminary remarks:

Remark 1. *([15, Theorem 1.22]) Let (R, m) be a Noetherian local ring, M a finitely generated R-module and N an arbitrary R-module. Then $\operatorname{Coass}(M \otimes_R N) = \operatorname{Supp} M \cap \operatorname{Coass} N$.*

proof. Note that $(M \otimes_R N)^{\mathrm{v}} \cong \operatorname{Hom}_R(M, N^{\mathrm{v}})$. Therefore

$$\begin{aligned} \operatorname{Coass}(M \otimes_R N) &= \operatorname{Ass}(\operatorname{Hom}_R(M, N^{\mathrm{v}}) \\ &= \operatorname{Supp} M \cap \operatorname{Ass} N^{\mathrm{v}} \qquad \text{(e.g., [2, IV.1.4, Prop. 10])} \\ &= \operatorname{Supp} M \cap \operatorname{Coass} N. \end{aligned}$$

Remark 2. *Let R be a local ring of dimension d, I an ideal of R and M an R-module. Then $H^d_I(M) \cong M \otimes_R H^d_I(R)$.*

proof. Since $H^d_I(-)$ is a right exact functor, this remark is an immediate consequence of Watts' Theorem [11, Theorem 3.33]. Here is a more direct proof: since R is local, there exist elements $\underline{x} = x_1, \ldots, x_d \in I$ which generate I up to radical. Then $H^i_I(M) = H^i_{(\underline{x})}(M)$ for all i. Using the Čech complex to compute $H^d_{(\underline{x})}(R)$, we see there is an exact sequence

$$\oplus_i R_{x_1 \cdots \hat{x}_i \cdots x_d} \longrightarrow R_{x_1 \cdots x_d} \longrightarrow H^d_{(\underline{x})}(R) \longrightarrow 0.$$

Tensoring this sequence with M, we get the exact sequence

$$\oplus_i M_{x_1 \cdots \hat{x}_i \cdots x_d} \xrightarrow{f} M_{x_1 \cdots x_d} \longrightarrow M \otimes_R H^d_{(\underline{x})}(R) \longrightarrow 0.$$

Since $\operatorname{coker} f = H^d_{(\underline{x})}(M)$, we see that $H^d_I(M) \cong M \otimes_R H^d_I(R)$.

The next result is essentially a module version of [12, Theorem 3.4] combined with [15, Theorem 1.16]. As in [12], we make repeated use of the Hartshorne-Lictenbaum vanishing theorem (HLVT): if (R, m) is a complete local ring of dimension d and I is an ideal of R, then $H^d_I(R) \neq 0$ if and only if $I + p$ is m-primary for some prime ideal p such that $\dim R/p = d$ ([5, Theorem 3.1]).

Lemma 3. *Let (R, m) be a complete Noetherian local ring, I an ideal of R and M a finitely generated R-module of dimension n. Then*

$$\operatorname{Coass} H^n_I(M) = \{p \in \mathrm{V}(\operatorname{Ann}_R M) \mid \dim R/p = n \text{ and } \sqrt{I + p} = m\}.$$

proof. Let $S = R/\operatorname{Ann}_R M$ and $E_S = \operatorname{Hom}_R(S, E)$ the injective hull of the residue field of S. Observe that

$$\begin{aligned} \operatorname{Hom}_S(H^n_{IS}(M), E_S) &\cong \operatorname{Hom}_R(H^n_I(M), E_S) \\ &\cong \operatorname{Hom}_R(H^n_I(M) \otimes_R S, E) \\ &\cong \operatorname{Hom}_R(H^n_I(M), E). \end{aligned}$$

Consequently, $\operatorname{Coass}_R H^n_I(M) = \pi(\operatorname{Coass}_S H^n_{IS}(M))$ where $\pi\colon \operatorname{Spec} S \to \operatorname{Spec} R$. Thus, we may assume that $\operatorname{Ann}_R M = 0$ and $n = \dim R$. By Remarks 1 and 2, we have $\operatorname{Coass} H^n_I(M) = \operatorname{Coass}(M \otimes_R H^n_I(R)) = \operatorname{Coass} H^n_I(R)$, so it is enough to prove the result in the case $M = R$. By HLVT, both sets are empty if $H^n_I(R) = 0$, so assume that $H^n_I(R) \neq 0$. Let $q \in \operatorname{Coass} H^n_I(R)$. By the remark, $q \in \operatorname{Coass}(R/q \otimes_R H^n_I(R))$. In particular, $R/q \otimes_R H^n_I(R) \cong H^n_I(R/q) \neq 0$. Thus, $n = \dim R/q$ and $I + q$ is m-primary (by HLVT). Now suppose $\dim R/q = n$ and $\sqrt{I + q} = m$. By reversing the above argument we get that $R/q \otimes_R H^n_I(R) \neq 0$. Let $p \in \operatorname{Coass}(R/q \otimes_R H^n_I(R))$. By Remark 1, $p \supseteq q$ and $p \in \operatorname{Coass} H^n_I(R)$. We've already shown that every coassociated prime of $H^n_I(R)$ is a minimal prime of R. Hence $p = q$ and $q \in \operatorname{Coass} H^n_I(R)$, which completes the proof.

We now show that $H^{\dim M}_I(M)$ is I-cofinite:

Theorem 3. *Let (R, m) be Noetherian local ring, I an ideal of R and M a finitely generated R-module of dimension n. Then $H^n_I(M)$ is I-cofinite. In fact, $\operatorname{Ext}^i_R(R/I, H^n_I(M)$ has finite length for all i.*

proof. By Lemma 2 we may assume that R is complete. Let $\operatorname{Coass} H^n_I(M) = \{p_1, \dots, p_k\}$. Since $H^n_I(M)$ is Artinian (see [12, Theorem 3.3], $H^n_I(M)^\vee$ is finitely generated. Hence $\operatorname{Supp} H^n_I(M)^\vee = \mathrm{V}(p_1 \cap \cdots \cap p_k)$. By Matlis duality, $\operatorname{Ext}^i_R(R/I, H^n_I(M))$ has finite length if and only if $\operatorname{Ext}^i_R(R/I, H^n_I(M))^\vee \cong \operatorname{Tor}^R_i(R/I, H^n_I(M)^\vee)$ ([11, Theorem 11.57]) has finite length. Since $\operatorname{Tor}^R_i(R/I, H^n_I(M)^\vee)$ is finitely generated, it is enough to show its support is contained in $\{m\}$. But

$$\begin{aligned} \operatorname{Supp} \operatorname{Tor}^R_i(R/I, H^n_I(M)^\vee) &\subseteq V(I) \cap \operatorname{Supp} H^n_I(M)^\vee \\ &= \mathrm{V}(I) \cap V(p_1 \cap \cdots \cap p_k) \\ &= V(I + (p_1 \cap \cdots \cap p_k)) \\ &= \{m\} \qquad \text{(by Lemma 3).} \end{aligned}$$

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