Cohen-Macaulay dimension for coherent rings

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Coherent rings

Joint work with Becky Egg.

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Examples include:

- All Noetherian rings.
- Semi-heriditary rings (e.g., valuation domains)
- Polynomial rings in any number of variables (finite or infinite) with coefficients from the above rings.
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- Quotients of such rings by finitely generated ideals.

Remark

If R is coherent then any f.p. R-module has a free resolution in which the free modules all have finite rank.

Sarah's Question

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Question (S. Glaz, 1994)

Does there exists a workable definition of Cohen-Macaulay (CM) for commutative rings which extends the usual definition in the Noetherian case and such that every coherent regular ring is CM?

One answer

In 2007, Tracy Hamilton and I gave a definition for CM which meets Glaz's requirements.

We use Čech cohomology to define sequences of elements from the ring which, in the Noetherian case, would mean they generate ideals of the principal class. A ring is then CM if all such sequences are regular sequences.

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In 2007, Tracy Hamilton and I gave a definition for CM which meets Glaz's requirements.

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First, the good news:

- Coherent regular rings are (locally) CM.
- Zero-dimensional rings and one-dimensional domains are CM.
- If S is a faithfully flat R-algebra and S is CM, so is R.
- If R is an excellent domain of characteristic $p > 0$, then R^+ is CM.
- Rings of invariants of certain finite groups acting on coherent regular rings are CM (Asgharzadeh and Tousi, 2009).

One answer (continued)

The bad news:

- If R is CM and x is a nzd, $R/(x)$ need not be CM.
- If R is CM and p is a prime ideal, it is unknown if R_p is CM.
- If R is CM and t is an indeterminate, it is unknown if $R[t]$ is CM.

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In 2009, Livia Hummel and I developed a theory for coherent Gorenstein rings which seemed to work much better. For this, we extend the work of Auslander and Bridger on Gorenstein dimension to coherent rings.

Grade

Let I be a finitely generated ideal of a coherent ring R and M a f.p. R-module such that $IM \neq M$. Define

$$\operatorname{grade}_I M := \inf\{n \mid \operatorname{Ext}^n_R(R/I, M) \neq 0\}.$$

If (R, m) is quasi-local then define

$$\operatorname{depth} M := \sup\{\operatorname{grade}_I M \mid I \subseteq m, I \text{ f.g}\}.$$

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Remark

It is possible for depth $M > 0$, yet m consist of zero-divisors on M. This can be corrected by passing to a faithfully flat extension of R.

Semidualizing modules

Let (R, m) be a quasi-local coherent ring. A f.p. R-module K is called a *semidualizing* module for R if

- The natural map $R \to \operatorname{Hom}_R(K, K)$ is an isomorphism.
- $\operatorname{Ext}^i_R(K, K) = 0$ for all $i > 0$.

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Note that R is a semidualizing module.

Totally K-reflexive modules

Let K be a semidualizing R-module and M a f.p. R-module M. Let $M^\dagger := \mathrm{Hom}_R(M, K)$. We say M is *totally K-reflexive* if

- $\mathrm{Ext}^i_R(M, K) = 0$ for $i > 0$.
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We note that $R, K \in G_K(R)$ and that $G_K(R)$ is closed under direct sums, summands, and K-duals.

G_K-dimension

Let R be coherent and M a nonzero f.p. R-module. A G_K-resolution of M of length n is an acyclic complex

$$\mathbf{G}: \qquad 0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to 0$$

such that

- $G_i \in G_K(R)$ for all i.
- $G_n \neq 0$.
- $H_0(\mathbf{G}) \cong M$.

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We set G_K-dim M to be the length of the smallest finite G_K-resolution of M, assuming one exists. Otherwise, we set G_K-dim $M = \infty$.

Auslander-Bridger formula

Theorem (A-B 1969, Gerko 2001, Egg-M. 2014)

Let (R, m) be a quasi-local coherent ring, K a semidualizing module, and M a nonzero f.p. module. Then if G_K-dim $M < \infty$ then

$$G_K\text{-}dim\ M + \text{depth}\ M = \text{depth}\ R.$$

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Following Gerko, we set CMdim $M := \inf\{G_K\text{-dim}\ M \otimes_R S\}$, where the infimum is taken over all faithfully flat quasi-local extensions S of R and semidualizing modules K for S. So if CMdim $M < \infty$, we have

$$\text{CMdim}\ M + \text{depth}\ M = \text{depth}\ R.$$

A new definition of CM

Theorem (Gerko, 2001)

Let (R, m, k) be a local ring. The following are equivalent:

- *R is CM.*
- *CMdim $M < \infty$ for all nonzero f.g. R-modules.*
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Definition (Egg-M., 2014)

Let (R, m) be a quasi-local coherent ring. We define R to be GCM (CM in the sense of Gerko) if CMdim $M < \infty$ for all nonzero f.p. R-modules.

Results on GCM, part I

Theorem (Egg-M., 2014)

Let (R, m) be a quasi-local coherent ring. The following hold:

- *If R is Gorenstein then R is GCM.*
- *If R is GCM and $x \in m$ is a nzd then $R/(x)$ is GCM.*
- *If R is GCM and p is a prime ideal then R_p is GCM.*
- *If t is an indeterminate and $R[t]$ is coherent, then R is GCM if and only if $R[t]$ is GCM.*

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There are several things we don't know: Does the converse to the second item hold? Does GCM imply CM (in the sense of Hamilton-M.)? Are zero-dimensional rings GCM?

Results on GCM, part II

Theorem (Egg-M., 2014)

Let S be a quasi-local coherent Gorenstein ring of finite depth. Let I be a finitely generated ideal of S and let $R = S/I$. Then:

- *R is GCM if and only* $\operatorname{depth} R = \operatorname{depth} S - \operatorname{grade} I$.
- *If R is GCM then $K = \operatorname{Ext}_S^t(R, S)$ ($t = \operatorname{grade} I$) is a semidualizing module for R and G_K-dim $M < \infty$ for all nonzero f.p. R-modules M.*

Results on GCM, part III

Theorem (Egg -M., 2014)

Let $R = S/I$ as in the previous theorem.

- *If $x \in m$ is a nzd on R and $R/(x)$ is GCM, then R is GCM.*
- *If $\dim R = 0$ then R is GCM.*
- *R is GCM then R is CM (in the sense of Hamilton-M.).*

An application of Gruson's Theorem

Proposition

Let (R, m) be a quasi-local coherent ring, K a semidualizing module for R, and $x \in m$ a nzd on R. Then x is a nzd on K.

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Proposition

Let (R, m) be a quasi-local coherent ring, K a semidualizing module for R, and $x \in m$ a nzd on R. Then x is a nzd on K.

Proof: K is a f.g. faithful R-module. By Gruson, there exists a finite filtration

$$0 = q_t \subset q_{t-1} \subset \cdots \subset q_0 = R/(x)$$

where for each i, q_i/q_{i+1} is quotient of a direct sum of some number (possibly infinite) of copies of K.

Proof, continued

Hence, there is an injection

$$0 \to \mathrm{Hom}_R(q_i/q_{i+1}, K) \to \mathrm{Hom}(\oplus_i K, K) \cong \prod_i R.$$

Proof, continued

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$$0 \to \mathrm{Hom}_R(q_i/q_{i+1}, K) \to \mathrm{Hom}(\oplus_i K, K) \cong \prod_i R.$$

As $xq_i = 0$ for all i and x is a nzd on R and hence on $\prod_i R$, we see that $\mathrm{Hom}_R(q_i/q_{i+1}, K) = 0$ for all i.

Proof, continued

Hence, there is an injection

$$0 \to \operatorname{Hom}_R(q_i/q_{i+1}, K) \to \operatorname{Hom}(\oplus_i K, K) \cong \prod_i R.$$

As $xq_i = 0$ for all i and x is a nzd on R and hence on $\prod_i R$, we see that $\operatorname{Hom}_R(q_i/q_{i+1}, K) = 0$ for all i.

Using the short exact sequences $0 \to q_{i+1} \to q_i \to q_i/q_{i+1} \to 0$, we obtain that $\operatorname{Hom}_R(q_i, K) = 0$ for all i. As $q_0 = R/(x)$, we see that $\operatorname{Hom}_R(R/(x), K) = 0$, which implies x is a nzd on K.

Thank you!