The Binomial Theorem without middle terms: Putting prime numbers to work in algebra

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Rings!

RING

THEORY

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General rule:

Divide by 12 and take remainder.

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$$7 - 13 = -6 = 10$$
$$5 \times 7 = 35 = 3$$

Similarly for $\mathbb{Z}_n$ for any integer $n \geq 1$.

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One can also subtract in rings, but not necessarily divide.
For example, in $\mathbb{Z}_{12}$ we have: $6 \times 3 = 6 \times 9$.
However: $3 \neq 9$ in $\mathbb{Z}_{12}$.

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We can also consider all power series in x with coefficients from R:

$$R[[x]] := \{\sum_{i=0}^{\infty} a_i x^i \mid a_i \in R \ \forall \ i\}.$$

Polynomials and power series (cont.)

We can iterate these processes:

- $R[x, y] := (R[x])[y]$
- $R[x, y, z] := (R[x, y])[z]$
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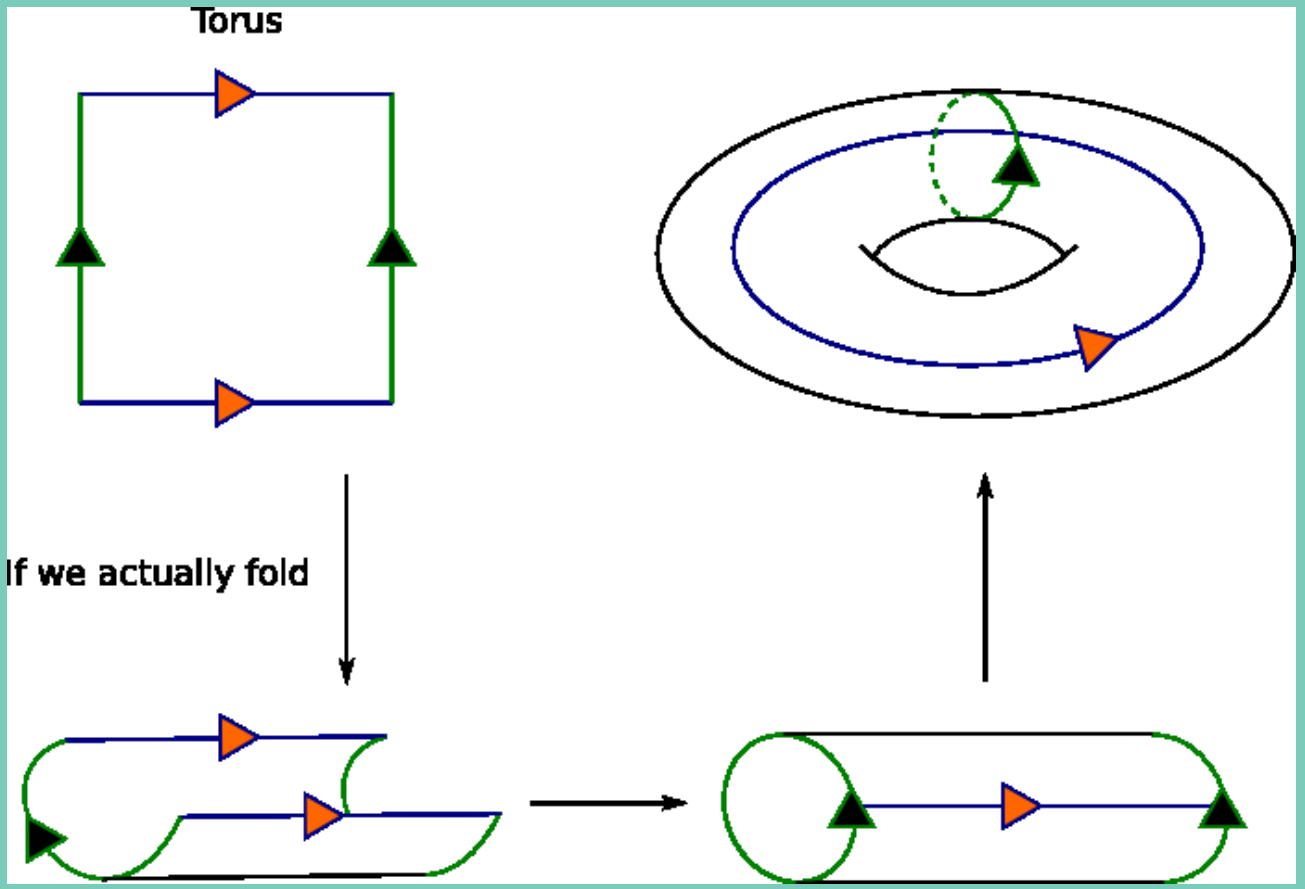
For example, in $\mathbb{Z}[[x, y]]$, we have:

$$(1 - xy) \cdot (1 + xy + x^2y^2 + \cdots + x^iy^i + \cdots) = 1.$$

Quotient spaces



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To invert an element $a \in R$, just consider the ring:

$$R[x]/(ax - 1).$$

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- Number theory, algebraic geometry, invariant theory

The Binomial Theorem

Let p be a (positive) prime integer. For the remainder of this talk, we'll restrict our attention to rings of the form

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Key Observation

The set

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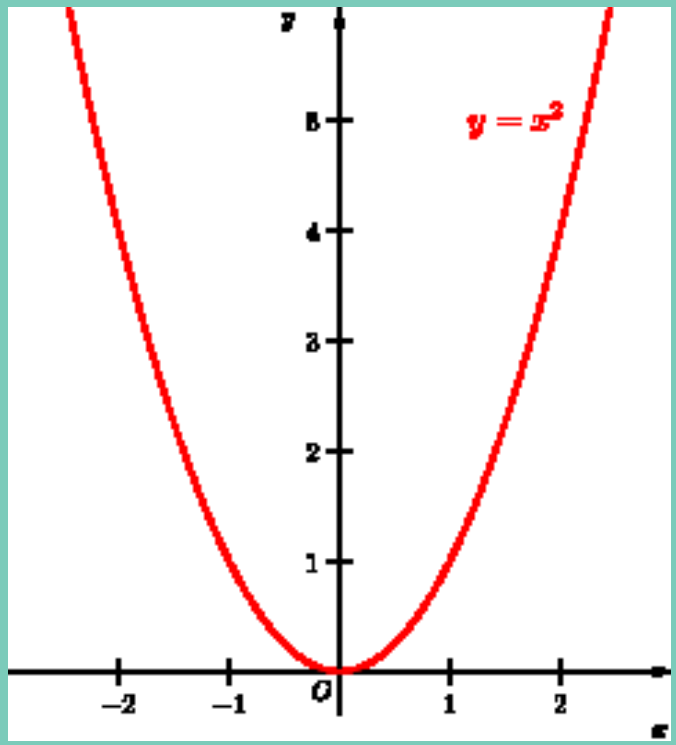
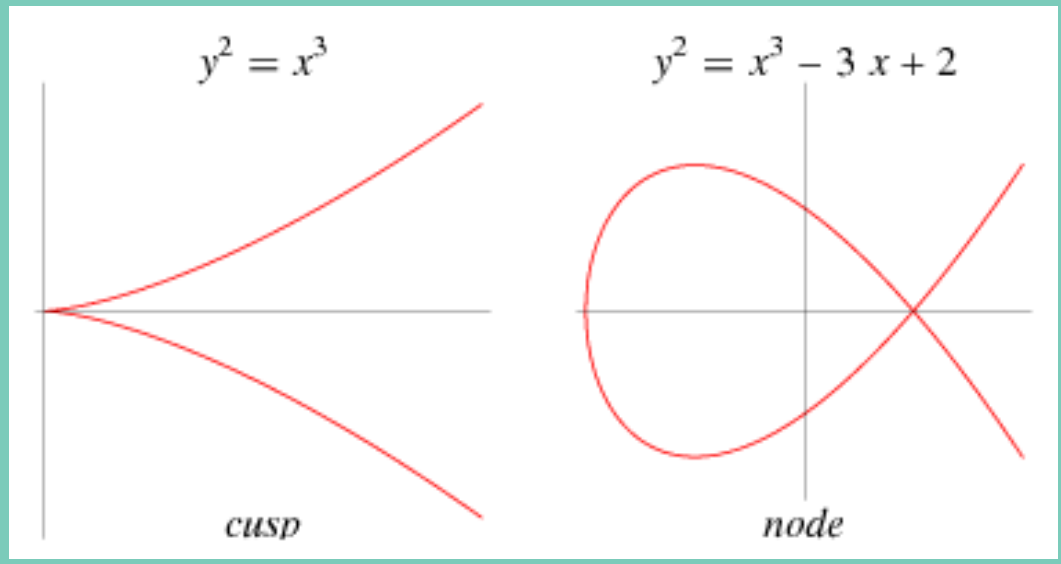
The relationship between the ring R and the subring R^p provides important clues about the structure of R.

Curve singularities





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For the node:

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where $c=\sqrt{x+1}=1+\frac{1}{2}x-\frac{1}{8}x^2+\cdots$.

Smoothness

Definition

Let $f(x, y) = 0$ define a curve C. We say C is smooth or nonsingular at the origin if its coordinate ring $k[[x, y]]/(f)$ is isomorphic to a polynomial ring in one variable. Otherwise, C is singular at the origin.

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Remark

If $k = \mathbb{R}$ then C is smooth at $(0, 0)$ if and only if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ don't both vanish at the origin. That is, C is smooth at $(0, 0)$ if and only if there is a unique well-defined tangent line to C at the origin.

An illustrative example

Question

Given a ring $R = \mathbb{Z}_p[[x_1, \ldots, x_d]]/(f_1, \ldots, f_r)$, how can we decide if R is smooth at the origin? That is, how can we tell if $R \cong \mathbb{Z}_p[[y_1, \ldots, y_s]]$?

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Consider $R = \mathbb{Z}_p[[x]]$. Then $R^p = \mathbb{Z}_p[[x^p]]$.

Note that every element in $f \in R$ can be written uniquely in the form:

$$f = c_0 + c_1 x + c_2 x^2 + \cdots + c_{p-1} x^{p-1}$$

for some $c_0, \ldots, c_{p-1} \in R^p$.

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For example, let $p = 3$ and $f = 2 + x + 2x^3 + x^4 + 2x^5 + x^7$. Then

$$f = (2 + 2x^3) \cdot 1 + (1 + x^3 + x^6)x + (2x^3)x^2.$$

An illustrative example, cont.

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$$t^6 \cdot 1 + t^4 \cdot t^2 = 0.$$

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Theorem (Avramov-Hochster-Iyengar-Yao, 2012)

If there exists any nonzero R-module which has a finite basis over R^p then $R \cong \mathbb{Z}_p[[x_1, \ldots, x_r]]$.

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