A THEOREM OF GRUSON

BRIAN JOHNSON

1. Gruson's Theorem

This is based on a proof of the result given in [4].

Let A be a commutative ring, M a finitely generated A-module, and

$$F \xrightarrow{\phi} A^n \longrightarrow M \longrightarrow 0$$

a presentation of M, where F is a free module of arbitrary rank. Let U denote the matrix (with n rows but possibly infinitely many columns) representing ϕ with respect to some fixed bases for F and A^n. For $t \geq 1$ we let $I_t(\phi)$ denote the ideal of A generated by all t-sized minors U. (This ideal is independent of the choice of bases.) By convention, $I_t(\phi) = R$ for $t \leq 0$ and $I_t(\phi) = 0$ if t exceeds n or the rank of F. For an integer i, we let $F_i(M) := I_{n-i}(\phi)$. It is a result of Fitting that $F_i(M)$ does not depend on the presentation of M (e.g., [3, Theorem 1, pg. 58]). If the ring A is not clear from the context, we write $F_i^A(M)$ for $F_i(M)$.

A projective A-module is said to have *constant rank* r if $P_q \cong A_q^r$ for every prime ideal q.

Lemma 1.1. *Suppose E is a finitely generated A-module. Then E is projective of constant rank r if and only if $F_{r-1}(E) = 0$ and $F_r(E) = A$.*

Proof. See [1, Proposition 1.4.9] for the case E is finitely presented. We now prove the general case. If E is finitely generated and projective, then it is finitely presented. Conversely, let

$$F \xrightarrow{\phi} A^n \longrightarrow E \longrightarrow 0$$

be a presentation for E, and let U be a matrix representing ϕ with respect to some choice of bases. Since $F_r(E) = A$, there exists a submatrix B of U with n rows and a finite number of columns such that $I_{n-r}(B) = I_{n-r}(U) = A$. We claim that $\operatorname{im} B = \operatorname{im} U$, which would imply E is finitely presented. Let B' be the matrix obtained by adjoining to B an arbitrary column of U. It suffices to show $\operatorname{im} B = \operatorname{im} B'$. Since $\operatorname{im} B \subseteq \operatorname{im} B'$, it suffices to prove this locally at any maximal ideal m. So assume A is local. Certainly, we have $I_{n-r}(B) = I_{n-r}(B') = A$ and $I_{n-r+1}(B) = I_{n-r+1}(B') = 0$ (since

Date: Summer 2009.

$I_{n-r+1}(U) = 0$). Let C and C' be the cokernels of B and B', respectively. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{im} B & \longrightarrow & A^n & \longrightarrow & C & \longrightarrow & 0 \\ & & \Big\downarrow i & & \Big\downarrow = & & \Big\downarrow \alpha & & \\ 0 & \longrightarrow & \operatorname{im} B' & \longrightarrow & A^n & \longrightarrow & C' & \longrightarrow & 0, \end{array}$$

where i is the natural inclusion map. By the finitely presented case, C and C' are both free of rank r. By the Snake Lemma, α is surjective and hence an isomorphism. Again by the Snake Lemma, $\operatorname{coker} i \cong \ker \alpha = 0$ and hence i is surjective. Thus, $\operatorname{im} B = \operatorname{im} B'$. □

Lemma 1.2. *Suppose A is a commutative ring and P is a finitely generated projective A-module. If P has constant (positive) rank then there exists d such that $P^d \cong A \oplus Q$ for some A-module Q.*

Proof. Since P is finitely generated and projective, P is finitely presented. If A is Noetherian and finite dimensional, we are done by a theorem of Serre [2, Corollary 2.5] which we prove in Section 2. If not, consider a presentation of P:

$$A^m \xrightarrow{\phi} A^n \longrightarrow P \longrightarrow 0.$$

Let $U = (u_{ij})$ be a matrix representing ϕ. Let T be the prime subring of A, and let $D_1, \dots, D_s$ be the $r \times r$ minors of U (with r as above). Then there exists $b_1, \dots, b_s \in A$ such that $b_1 D_1 + \cdots + b_s D_s = 1$. Set

$$S = T[\{u_{ij}\}, b_1, \dots, b_s] \subseteq A.$$

Now, since $T = \mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$, T is Noetherian and finite dimensional. Thus, S is also Noetherian and finite dimensional. Let $M = \operatorname{coker}(S^m \xrightarrow{U} S^n)$. Then $F_r^S(M) = S$ and $F_{r-1}^S(M) = 0$ (where the superscript S simply denotes the fact we are considering the Fitting invariants as ideals of S). By 1.1, M is a projective S-module. Then by the Noetherian case,

$$M^d \cong S \oplus N,$$

for some d. Applying $A \otimes_S -$ to

$$S^m \longrightarrow S^n \longrightarrow M \longrightarrow 0,$$

we see $A \otimes_S M \cong P$. But

$$P^d \cong A \otimes_S M^d \cong (A \otimes_S S) \oplus (A \otimes_S N) \cong A \oplus (A \otimes_S N),$$

and we are done. □

Theorem 1.3 (Gruson)**.** *Let E be a finitely generated faithful module over the commutative ring A. Then every A-module admits a finite filtration of submodules whose factors are quotients of direct sums of copies of E.*

Proof. We first claim it is enough to show this for the ring A itself. Indeed, suppose $0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_k = A$ is a sequence of ideals of A such that for each $i = 1, \dots, k$ there is a surjection

$$\phi_i : E^{(L_i)} \longrightarrow I_i / I_{i-1}.$$

Then suppose M is any A-module with presentation $A^{(L)} \xrightarrow{\phi} M \longrightarrow 0$. consider the series of submodules of M given by $\phi(I_i^{(L)})$. First, notice that

$$0 = \phi(I_0^{(L)}) \subseteq \phi(I_1^{(L)}) \subseteq \cdots \subseteq \phi(I_k^{(L)}) = \phi(A^{(L)}) = M$$

is indeed a filtration of M. Then it will suffice to show there is a surjection $E^{(K)} \twoheadrightarrow \phi(I_i^{(L)})/\phi(I_{i-1}^{(L)})$. Define a map

$$\psi_i : I_i^{(L)}/I_{i-1}^{(L)} \longrightarrow \phi(I_i^{(L)})/\phi(I_{i-1}^{(L)})$$

by $x + I_{i-1}^{(L)} \mapsto \phi(x) + \phi(I_{i-1}^{(L)})$. One checks easily that this is a well-defined homomorphism. Note that every element of $\phi(I_i^{(L)})/\phi(I_{i-1}^{(L)})$ is of the form $\phi(x) + \phi(I_{i-1}^{(L)})$ for some $x \in I_i^{(L)}$, so ψ_i is surjective. Thus, we have the desired surjection

$$E^{(K)} = (E^{(L_i)})^{(L)} \twoheadrightarrow I_i^{(L)}/I_{i-1}^{(L)} \twoheadrightarrow \phi(I_i^{(L)})/\phi(I_{i-1}^{(L)}).$$

We now proceed in the case of the ring A by using the Fitting invariants $F_i := F_i(E)$ of the module E. Let

$$F \xrightarrow{\phi} A^n \longrightarrow E \longrightarrow 0$$

be a free presentation of E and let $U = (u_{ij})$ be a matrix representing ϕ. Define r to be the smallest integer such that $F_r \neq 0$, and m to be the smallest integer such that $F_m = A$. Note first that as $F_0 \subseteq \operatorname{Ann}_A E$, we must have $F_0 = 0$ since E is faithful. This implies that $r > 0$.

We'll now use induction on $d = m - r$. If $d = 0$, Lemma 1.1 implies E is projective of constant rank, and we use Lemma 1.2 to get ℓ such that $E^\ell \cong A \oplus Q$. Since every A-module is the quotient of a free module, we can clearly map a large enough direct sum of copies of E onto any A-module we please, and we're done in this case. Suppose $d > 0$ (i.e., $m > r$). Let $I = \operatorname{Ann}_A(E/F_rE)$ and consider the A/I-module E/IE. Suppose $\overline{x} \in \operatorname{Ann}_{A/I}(E/IE)$. Then $xE \subseteq IE$, and as $I = \{a \in A \mid aE \subseteq F_rE\}$, certainly $IE \subseteq F_rE$. But this implies that $xE \subseteq F_rE$, whence $x \in I$. Thus $\overline{x} = \overline{0}$, and E/IE is a faithful A/I-module. Also, notice that since $F_r \subseteq I$, we must have $F_r^{A/I}(E/IE) = 0$. Indeed, given the presentation for E above, we may mod out by I to obtain a presentation for E/IE, and we find the Fitting invariants are just $F_i^{A/I}(E/IE) = \frac{F_i(E)+I}{I}$. Thus, $F_r^{A/I}(E/IE) = 0$ and $F_m^{A/I}(E/IE) = A/I$, and we may apply the induction hypothesis to the A/I-module E/IE to get a finite filtration of A/I modules whose factors are quotients of direct sums of E/IE, say

$$I = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_k = A,$$

and the factors remain unchanged, as $A_i/A_{i-1} \cong (A_i/I)/(A_{i-1}/I)$. As these factors are quotients of direct sums of E/IE, we may compose with the natural map $E \twoheadrightarrow E/IE$, and we see that they are, in fact, quotients of direct sums of E as well. Now, all that remains is to find a finite filtration of I in which the factors have the desired property. We first claim that $I^n \subseteq F_r$. To see this, we use the fact that $\mathrm{Ann}_{A/F_r}(E/F_rE)^n \subseteq F_0^{A/F_r}(E/F_rE)$. But $F_0^{A/F_r}(E/F_rE) = \overline{0}$ as $F_0(E) = 0$. Thus, $I^n = \mathrm{Ann}_A(E/F_rE)^n \subseteq F_r$. Then we have the following filtration:

$$F_r \subseteq F_r + I^{n-1} \subseteq F_r + I^{n-2} \subseteq \cdots \subseteq F_r + I = I.$$

Note that $I(F_r + I^{n-i}) \subseteq F_r + I^{n-i+1}$, so $\dfrac{F_r + I^{n-i}}{F_r + I^{n-(i-1)}}$ is an A/I-module for each i. As above, these can be seen to be quotients of direct sums of E. So now it only remains to show that we may write F_r as a quotient of a direct sum of copies of E. Let $s := n - r$ and let D be the determinant of the $s \times s$ submatrix of U given by (u_{i_k,j_l}), where $1 \le k, l \le s$ and $i_k, j_l \in \{1, \ldots, n\}$. We will find a linear form on E such that its image contains D. Choose $t \in \{1, \ldots, n\}$ such that $t \neq i_k$ for any k (this is possible as $r > 0$). Consider the following linear form on A^n:

$$\underline{x} := \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \det \begin{pmatrix} u_{i_1,j_1} & \cdots & u_{i_1,j_s} & x_1 \\ \vdots & \vdots & \vdots & \vdots \\ u_{i_s,j_1} & \cdots & u_{i_s,j_s} & x_s \\ u_{t,j_1} & \cdots & u_{t,j_s} & x_t \end{pmatrix}.$$

Now, as $s + 1 = n - r + 1 = n - (r - 1)$, and $F_{r-1} = 0$, we have that the $(s+1) \times (s+1)$ minors of U must be zero. Thus, this form is trivial on the image of U. Indeed, if $\underline{x} \in \mathrm{im}\, U$, then $\underline{x}$ is an A-linear combination of the columns of U, and so the determinant will be an A-linear combination of $(s+1) \times (s+1)$ minors of U. By passing to the quotient mod the image of U, we obtain a linear form on $E = \mathrm{coker}\, U$. That is, let e_i be the standard ith basis vector. We have a map $\phi_D : E \longrightarrow R$, and note that for all i, $\phi(\overline{e_i})$ is an $(n - r) \times (n - r)$ minor of U, so $\phi_D : E \longrightarrow F_r(E)$. Finally, note

$$e_t \mapsto \det \begin{pmatrix} u_{i_1,j_1} & \cdots & u_{i_1,j_s} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ u_{i_s,j_1} & \cdots & u_{i_s,j_s} & 0 \\ u_{t,j_1} & \cdots & u_{t,j_s} & 1 \end{pmatrix} = D,$$

so we can map a copy of E onto each of the generators of F_r. □

A more general version of Gruson's Theorem is the following:

Corollary 1.4. *Let A be a commutative ring, M a finitely generated A-module, and $I = \operatorname{Ann}_A M$. Then for every A-module L such that $I^n L = 0$ for some n, there exists a finite filtration of submodules of L such that the factor modules are quotients of direct sums of copies of M.*

Proof. Consider the filtration

$$0 = I^n L \subseteq I^{n-1} L \subseteq \cdots \subseteq I^0 L = L.$$

Each of the factors in this filtration is an A/I-module. Since M is a finitely generated faithful A/I-module, each of the factors has a finite filtration whose factor modules are quotients of direct sums of M. Patching these filtrations together gives the desired finite filtration of L. □

A special case of this result is the following:

Corollary 1.5. *Let A be a Noetherian ring and L and M finitely generated A-modules. Suppose $\operatorname{Supp}_A L \subseteq \operatorname{Supp}_A M$. Then there exists a finite filtration of L whose factors are quotients of directs sums of copies of M.*

The following is a frequently used consequence of Gruson's Theorem:

Corollary 1.6. *Let A be a commutative ring, E a finitely generated faithful A-module, and M an arbitrary A-module. Then $E \otimes_A M = 0$ if and only if $M = 0$. That is, $E \otimes_A -$ is a faithful functor.*

Proof. Suppose $E \otimes_A M = 0$. It suffices to show that $N \otimes_A M = 0$ for any A-module N. Let $\lambda(N)$ denote the length of the shortest filtration of N such that the factor modules of the filtration are homomorphic images of direct sums of copies of E. If $\lambda(N) = 1$ then N is the homomorphic image of $E^{(L)}$ for some L. Since $E \otimes_A M = 0$, certainly $E^{(L)} \otimes_A M = 0$ and hence $N \otimes_A M = 0$. Suppose $\lambda(N) > 1$. Then there exists a short exact sequence $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ such that $\lambda(N')$ and $\lambda(N'')$ are less than $\lambda(N)$. By induction, $N' \otimes_A M = 0 = N'' \otimes_A M$. By the right exactness of tensor products, we see that $N \otimes_A M = 0$. □

2. A Theorem of Serre

This will be a proof of a theorem of Serre based on [2].

Throughout this section R denotes a commutative Noetherian ring.

Definition 2.1. Suppose M is a finitely generated R-module, $M' \subseteq M$ a submodule, and $p \in \operatorname{Spec} R$. We say M' is *w-fold basic* in M at p if

$$\mu((M/M')_p) \leq \mu(M_p) - w,$$

where $\mu(M)$ denotes the minimum number of generators of M. Equivalently, we have

$$\dim_{k(p)} \frac{M'_p + pM_p}{pM_p} \geq w,$$

where $k(p) := R_p/pR_p$.

Lemma 2.2. *Let M be a finitely generated R-module, let $p_1, \dots, p_n \in \operatorname{Spec} R$, and suppose the submodule $M' = \sum_{i=1}^{w} Rm_i$ is w_i-fold basic in M at p_i for each $i = 1, \dots, n$. Then there exist $r_1, \dots, r_{w-1} \in R$ such that $N = \sum_{i=1}^{w-1} R(m_i + r_i m_w)$ is $\min\{w-1, w_i\}$-fold basic in M at p_i for each $i = 1, \dots, n$.*

Proof. We use induction on n, the number of primes. The case $n = 0$ is vacuous. Suppose $n > 0$, and reorder the primes so that p_n is minimal among the p_i. That is, we have

$$p_1 \cap \cdots \cap p_{n-1} \not\subseteq p_n.$$

By induction, we may choose $r_1, \dots, r_{w-1}$ such that the lemma holds for $p_1, \dots, p_{n-1}$. Now, if N is $\min\{w-1, w_n\}$-fold basic in M at p_n, we are done. Suppose not. We claim this implies that $w - 1 \geq w_n$. To see this, note that since M' is w-generated, we must have $w_n \leq w$. Assuming $w_n = w$, we get

$$\min\{w-1, w_n\} = \min\{w-1, w\} = w - 1.$$

If M' is w-fold basic at p_n, it follows that N must be $(w-1)$-fold basic at p_n (by an elementary linear independence argument). Thus, if N is *not* $\min\{w-1, w_n\}$-fold basic at p_n, we must have that $w_n < w$, implying $w_n \leq w-1$, and hence $\min\{w-1, w_n\} = w_n$. Choose $r \in (p_1 \cap \cdots \cap p_{n-1}) \smallsetminus p_n$. We claim that

$$m_1 + r_1 m_w, \dots, m_{j-1} + r_{j-1} m_w, m_j + (r_j + r) m_w, \dots, m_{w-1} + r_{w-1} m_w$$

generate a w_n-fold basic submodule in M at p_n for some j. First, since N is not $\min\{w-1, w_n\}$-fold basic (recalling this implies $w_n \leq w-1$), we have

$$\dim_{k(p_n)} \frac{N_{p_n} + p_n M_{p_n}}{p_n M_{p_n}} \leq w - 1.$$

In particular, any set of $w-1$ elements of N must be linearly dependent in $(M/p_n M)_{p_n}$. Thus, for some $j \in \{2, \dots, w-1\}$, $m_j + r_j m_w$ is in the space spanned by the previous $j-1$ elements of the same form, say

$$m_j + r_j m_w = \sum_{i=1}^{j-1} s_i (m_i + r_i m_w) \tag{1}$$

in $(M/p_n M)_{p_n}$. We'll show this j satisfies our claim. Note that $\{m_i + r_i m_w\}_{i=1}^{w-1} \cup \{m_w\}$ generate M', which is w_n-fold basic in M at p_n. Consider the submodule L generated by

$$m_1 + r_1 m_w, \dots, m_{j-1} + r_{j-1} m_w, m_j + (r_j + r) m_w, \dots, m_{w-1} + r_{w-1} m_w.$$

Note that as $r \in p_1 \cap \cdots \cap p_{n-1}$, we have that $(L + p_i M)/p_i M = (N + p_i M)/p_i M$ for $i \in \{1, \dots, n-1\}$. Hence, L is $\min\{w-1, w_i\}$-fold basic in

M at p_i for $i = 1, \ldots, n-1$. We claim that $m_w \in L_{p_n}$, and therefore that $(M')_{p_n} = L_{p_n}$. By (1) we can write

$$rm_w = m_j + (r_j + r)m_w - \sum_{i=1}^{j-1} s_i(m_i + r_i m_w).$$

Further, r is a unit in $k(p)$, because $r \notin p_n$, so we can write m_w as a linear combination of the generators of L_{p_n}. Therefore, $L_{p_n} = (M')_{p_n}$, and the basicity of M' at p_n implies that L is also w_n-fold basic in M at p_n. □

Definition 2.3. Suppose M is a finitely generated R-module and t is a nonnegative integer. Define

$$J_t(M) = \{\sum_{M'} \operatorname{Ann}(M/M') \mid M' \subseteq M \text{ can be generated by } t \text{ elements}\}.$$

Remark 2.4. We note some properties.

(1) We have a chain

$$\operatorname{Ann}(M) = J_0(M) \subseteq J_1(M) \subseteq \cdots \subseteq J_n(M) \subseteq \cdots.$$

(2) If $\mu(M) \leq n$, Then $J_n(M) = J_{n+1}(M) = \cdots = R$.

(3) If p is a prime, then $\mu(M_p) > t$ if and only if $p \supseteq J_t(M)$.

Lemma 2.5. *Let the submodule M' be w-fold basic in M at each prime of height less than or equal to k. Then M' is w-fold basic in M at all but finitely many primes of height $k+1$.*

Proof. Since M is finitely generated, $J_t(M/M') \neq R$ for only finitely many t. Each of these has only finitely many primes minimal over it. We claim that if $\operatorname{ht} p = k+1$ and p is not minimal over any $J_t(M/M')$, then M' is w-fold basic in M at p. Suppose p is such a prime. Consider two cases.

Case 1: $p \not\supseteq J_0(M/M')$. Since $J_0(M/M') = \operatorname{Ann}(M/M')$, this implies that $(M/M')_p = 0$, so $\mu((M/M')_p) = 0$. Thus, to show that M' is w-fold basic at p, we only need to show that $0 \leq \mu(M_p) - w$. But, if $q \subsetneq p$ is a height k prime, we know

$$\begin{aligned} 0 &\leq \mu((M/M')_q) \\ &\leq \mu(M_q) - w \\ &\leq \mu(M_p) - w, \end{aligned}$$

and M' is w-fold basic at p.

Case 2: $p \supseteq J_0(M/M')$. In this case, there exists some t such that $J_t(M/M') \subseteq p$, but $J_{t+1}(M/M') \nsubseteq p$. Since p is *not* minimal over any $J_i(M/M')$, there is some prime q with height less than $k+1$ such that $p \supsetneq q \supseteq J_t(M/M')$. Then we have

$$\mu((M/M')_p) = \mu((M/M')_q) = t+1$$

by part (3) of Remark 2.4. We also have $\mu(M_p) \geq \mu(M_q)$ as noted earlier, and finally

$$\mu(M_q) - \mu((M/M')_q) \geq w$$

by assumption. Combining the previous observations, we have

$$\mu(M_p) - \mu((M/M')_p) \geq \mu(M_q) - \mu((M/M')_q) \geq w,$$

and M' is w-fold basic in M at p. □

Definition 2.6. Let M be an R-module and $N = Rn_1 + \cdots + Rn_s$ a submodule of M. Then $\{n_1, \ldots, n_s\}$ (the set of generators) is called *basic up to height* k if N (the submodule they generate) is $\min\{s, k - \operatorname{ht} p + 1\}$-fold basic in M at each prime p such that $\operatorname{ht} p \leq k$. If $k = \dim R$, we say N is *basic in* M. Further, if N is basic in M and $N = Rx$, we call x a *basic element* of M.

Lemma 2.7. *Let $N = Rm_1 + \cdots + Rm_n \subseteq M$ be basic up to height k with $n > 1$. Then there exist elements $r_1, \ldots, r_{n-1} \in R$ such that $R(m_1 + r_1 m_n) + \cdots + R(m_{n-1} + r_{n-1} m_n)$ is basic up to height k.*

Proof. By assumption, if p is of height $j - 1$, where $1 \leq j \leq k$, then $Rm_1 + \cdots + Rm_n$ is $\min\{n, k - j + 2\}$-fold basic in M at p. Thus, by Lemma 2.5, $Rm_1 + \cdots + Rm_n$ is $\min\{n, k - j + 2\}$-fold basic in M at q for all but finitely many primes q of height j. Note that we may replace $\{m_1, \ldots, m_n\}$ by $\{m_1 + r_1 m_n, \ldots, m_{n-1} + r_{n-1} m_n\}$ (for any $r_1, \ldots, r_n$) and sacrifice at worst one "basicity," since adjoining m_r to the latter set of generators gives a generating set for N. Thus, $R(m_1 + r_1 m_n) + \cdots + R(m_{n-1} + r_{n-1} m_n)$ is at least $\min\{n - 1, k - j + 1\}$-fold basic at q, which is what we require. Now, we are only concerned with finitely many j, and for each of these j, there are only finitely many primes of height j where $Rm_1 + \cdots + Rm_n$ is $\min\{n, k - j + 1\}$-fold basic, rather than the $\min\{n, k - j + 2\}$-fold basic obtained from the primes of height $j - 1$. So, for all of these finitely many primes, we may use Lemma 2.2 to find $r_1, \ldots, r_{n-1} \in R$ such that, at these primes, $R(m_1 + r_1 m_n) + \cdots + R(m_{n-1} + r_{n-1} m_n)$ is $\min\{n - 1, \min\{n, k - j + 1\}\}$-fold basic in M (for the primes of height j in the finite set). But

$$\min\{n - 1, \min\{n, k - j + 1\}\} = \min\{n - 1, k - j + 1\},$$

so we are done. □

Theorem 2.8. *If M, or more precisely, a set of generators for M, is basic up to height k, then there exists $m \in M$ such that Rm is basic up to height k.*

Proof. Suppose $M = Rm_1 + \cdots + Rm_n$. If $n > 1$, we need to show that we can replace $\{m_1, \ldots, m_n\}$ by a set with one fewer element but still basic up to height k. This is possible by the previous lemma. □

Definition 2.9. If R is an integral domain with quotient field K, then we define the rank of M to be

$$\operatorname{rank}_R M := \dim_K K \otimes_R M.$$

If R is not a domain, then we define the rank of M to be

$$\operatorname{rank}_R M := \min\{\operatorname{rank}_{R/p} M/pM \mid p \in \operatorname{Min} R\}.$$

Remark 2.10. If P is projective of constant rank r then $\operatorname{rank}_R P = r$.

Theorem 2.11 (Serre, 1958)**.** *Let R be a Noetherian ring of dimension d, and let P be a finitely generated projective R-module of rank greater than d. Then P has a free summand.*

Proof. Since $\mu(P) \geq \operatorname{rank}_{R/q}(P/qP)$ for any $q \in \operatorname{Spec} R$, we see that $\mu(P) \geq \operatorname{rank}_R(P)$. By assumption, $\operatorname{rank}_R(P) > d$, so we have that $\mu(P) \geq d+1$. We claim that any generating set for P is basic. That is, if $Rx_1 + \cdots + Rx_n = P$, we claim that $\{x_1, \ldots, x_n\}$ is basic. To prove this, we need to show that $\{x_1, \ldots, x_n\}$ is basic up to height d, or that P is $\min\{n, d-j+1\}$-fold basic at each prime of height j. We've already shown that $n \geq d+1$, and since $j \geq 0$, it follows that $\min\{n, d-j+1\} = d-j+1$. Let $q \in \operatorname{Spec} R$ have height j. To show that P is $(d-j+1)$-fold basic in P at q, we need to show that

$$\mu((P/P)_q) \leq \mu(P_q) - (d-j+1).$$

Clearly $\mu((P/P)_q) = 0$, so it is sufficient to show that $\mu(P_q) \geq d-j+1$. Since any height j prime must contain a height 0 prime (possibly itself if $j = 0$), say q', we have the following:

$$\mu(P_q) \geq \mu(P_{q'}) \geq d+1 \geq d-j+1,$$

where the middle inequality follows from our assumption on $\operatorname{rank} P$. Thus, $\{x_1, \ldots, x_n\}$ is a basic set, and by Theorem 2.8 there exists a single basic element $m \in P$. That $\{m\}$ is basic implies Rm is $\min\{1, d-j+1\}$-fold basic in P at each prime of height j. As $j \leq d$, we see $\min\{1, d-j+1\} = 1$. I.e.,

$$\mu((P/Rm)_q) \leq \mu(P_q) - 1.$$

That is, m is part of a minimal generating set for P_q. Since P_q is a free R_q-module, this implies that m is part of a basis for P_q, and hence the inclusion $R_q m \hookrightarrow P_q$ splits. As this holds for every prime q, we see that the inclusion $Rm \hookrightarrow P$ splits and $Rm \cong R$. □

Corollary 2.12. *Let R be a Noetherian ring of dimension d and P a projective module of positive rank. Then P^n has a free summand for all $n > d$.*

References

1. Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR MR1251956 (95h:13020)
2. E. Graham Evans and Phillip Griffith, *Syzygies*, London Mathematical Society Lecture Note Series, vol. 106, Cambridge University Press, Cambridge, 1985. MR MR811636 (87b:13001)
3. D. G. Northcott, *Finite free resolutions*, Cambridge University Press, Cambridge, 1976, Cambridge Tracts in Mathematics, No. 71. MR MR0460383 (57 #377)
4. Wolmer V. Vasconcelos, *Divisor theory in module categories*, North-Holland Publishing Co., Amsterdam, 1974, North-Holland Mathematics Studies, No. 14, Notas de Matemática No. 53. [Notes on Mathematics, No. 53]. MR MR0498530 (58 #16637)