

FROBENIUS AND HOMOLOGICAL DIMENSIONS OF COMPLEXES

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ABSTRACT. It is proved that a module M over a Noetherian local ring R of prime characteristic and positive dimension has finite flat dimension if $\mathrm{Tor}_i^R({}^eR, M) = 0$ for $\dim R$ consecutive positive values of i and infinitely many e . Here eR denotes the ring R viewed as an R -module via the e th iteration of the Frobenius endomorphism. In the case R is Cohen-Macaulay, it suffices that the Tor vanishing above holds for a single $e \geq \log_p e(R)$, where $e(R)$ is the multiplicity of the ring. This improves a result of D. Dailey, S. Iyengar, and the second author [6], as well as generalizing a theorem due to C. Miller [14] from finitely generated modules to arbitrary modules. We also show that if R is a complete intersection ring then the vanishing of $\mathrm{Tor}_i^R({}^eR, M)$ for single positive values of i and e is sufficient to imply M has finite flat dimension. This extends a result of L. Avramov and C. Miller [2].

1. INTRODUCTION

For the past half-century the Frobenius endomorphism has proved to be an effective tool for characterizing when a given finitely generated module M over a commutative Noetherian local ring R of prime characteristic p has certain homological properties. In 1973 Peskine and Szpiro [16] proved that if a finitely generated module M has finite projective dimension then $\mathrm{Tor}_i^R({}^eR, M) = 0$ for all positive integers i and e , where eR denotes the ring R viewed as an R -module via the e th iteration of the Frobenius endomorphism. Shortly thereafter, Herzog [8] proved the converse: In fact, he showed that if $\mathrm{Tor}_i^R({}^eR, M) = 0$ for all $i > 0$ and infinitely many e then M has finite projective dimension. Some twenty years later, Koh and Lee [10] established a stronger version of Herzog's result: If for some e sufficiently large $\mathrm{Tor}_i^R({}^eR, M) = 0$ for $\mathrm{depth} R + 1$ consecutive positive values of i then M has finite projective dimension. Later, Miller [14] showed that if R is CM of positive dimension then $\dim R$ consecutive vanishings of $\mathrm{Tor}_i^R({}^eR, M)$ for some e sufficiently large implies M has finite projective dimension.

One may ask to what extent do the above results hold for arbitrary (i.e., not necessarily finitely generated) modules. As Tor detects flatness rather than projectivity, we seek conditions which imply a given module has finite flat dimension. (It is a deep result of Jensen [9] and Raynaud and Gruson [17] that, in the case R has finite Krull dimension, a module M has finite flat dimension if and only if it has finite projective dimension. We choose not to make use of this result in this paper, however.) In [15], the second author together with M. Webb proved the analogue

Date: April 2, 2019.

2010 Mathematics Subject Classification. 13D05; 13D07, 13A35.

Key words and phrases. Frobenius endomorphism, flat dimension, injective dimension, complete intersection.

of Peskine and Szpiro's result for modules of finite flat dimension; that is, if M has finite flat dimension then $\mathrm{Tor}_i^R({}^eR, M) = 0$ for all positive integers i and e . Further, it was shown that the analogue of Herzog's result holds for arbitrary modules as well. Subsequently, Dailey, Iyengar and the second author [6] showed that if $\mathrm{Tor}_i^R({}^eR, M) = 0$ for $\dim R + 1$ consecutive positive values of i and infinitely many e , then M has finite flat dimension. The question remains whether, as in the results of Koh, Lee, and Miller, one can get by with fewer consecutive vanishings for arbitrary modules when $\dim R > 0$.

In the present paper, we show that in fact $\dim R$ consecutive vanishings of $\mathrm{Tor}_i^R({}^eR, M)$ for positive values of i and infinitely many e is sufficient to prove that M has finite flat dimension if $\dim R > 0$; if R is Cohen-Macaulay, it suffices to show these vanishings hold for some e greater than the multiplicity of the ring.

We also prove in the case R is a local complete intersection ring that the vanishing of $\mathrm{Tor}_i^R({}^eR, M)$ for some positive integers i and e imply that M has finite flat dimension. This generalizes a result of Avramov and Miller [2], who established this for finitely generated modules. We also show that all of the above results are valid for complexes. The following theorem summarizes our main results:

Theorem 1.1. *Let (R, \mathfrak{m}) be a local ring of prime characteristic p and positive dimension d . Let M be an R -complex such that $H_*(M)$ is bounded above. The following are equivalent:*

- (a) M has finite flat dimension;
- (b) There exists $t > \sup H_*(M)$ such that $\mathrm{Tor}_i^R({}^eR, M) = 0$ for $t \leq i \leq t + d - 1$ and infinitely many e .

If R is Cohen-Macaulay, condition (a) is equivalent to:

- (c) There exists $t > \sup H_*(M)$ such that $\mathrm{Tor}_i^R({}^eR, M) = 0$ for $t \leq i \leq t + d - 1$ for some $e \geq \log_p e(R)$, where $e(R)$ denotes the multiplicity of R .

If R is a local complete intersection of arbitrary dimension, then condition (a) is equivalent to:

- (d) $\mathrm{Tor}_i^R({}^eR, M) = 0$ for some $i > \sup H_*(M)$ and some $e > 0$.

Analogous results hold for $\mathrm{Ext}_R^i({}^eR, M)$ and injective dimension in the case the Frobenius endomorphism is finite. In fact, with the exception of the proof of (d) implies (a), our method of proof is to first establish the results for injective dimension and then deduce the corresponding statements for flat dimension using standard arguments.

2. PRELIMINARIES

Throughout this paper (R, \mathfrak{m}, k) will denote a commutative Noetherian local ring with maximal ideal \mathfrak{m} and residue field k . In the case R has prime characteristic p , we let $f : R \rightarrow R$ denote the Frobenius endomorphism; i.e., $f(r) = r^p$ for every $r \in R$. For an integer $e \geq 1$ we let eR denote the ring R viewed as an R -algebra via f^e ; i.e., for $r \in R$ and $s \in {}^eR$, $r \cdot s := f^e(r)s = r^{p^e}s$. If eR is finitely generated as an R -module for some (equivalently, all) $e > 0$, we say that R is F -finite.

We refer the reader to [3] for terminology and conventions regarding complexes. If M is an R -complex, we write M_* (respectively, M^*) to emphasize when we are indexing M homologically (respectively, cohomologically). It will occasionally be

useful to work in the derived category of R , which will be denoted by $D(R)$. We use the symbol ‘ \simeq ’ to denote an isomorphism in $D(R)$.

We first establish how the R -algebra eR (i.e., restriction of scalars) behaves with respect to flat extensions. Much of this is folklore, but we include it for the reader’s convenience.

Lemma 2.1. *Consider a commutative square of ring homomorphisms:*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

where B is flat over A , and D is flat over C . Then for any A -complex M and any C -complex N one has for each i an isomorphism of D -modules

$$\mathrm{Tor}_i^A(M, N) \otimes_C D \simeq \mathrm{Tor}_i^B(M \otimes_A B, N \otimes_C D).$$

Proof. We have the following isomorphisms in $D(D)$:

$$\begin{aligned} (M \otimes_A^{\mathbf{L}} N) \otimes_C D &\simeq (M \otimes_A^{\mathbf{L}} D) \otimes_C^{\mathbf{L}} N \\ &\simeq M \otimes_A^{\mathbf{L}} (B \otimes_B^{\mathbf{L}} D) \otimes_C^{\mathbf{L}} N \\ &\simeq (M \otimes_A B) \otimes_B^{\mathbf{L}} (D \otimes_C N). \end{aligned}$$

Taking homology and using that $-\otimes_C D$ is exact gives the desired result. \square

Corollary 2.2. *Suppose R has prime characteristic and S is a flat R -algebra. Let M be an R -complex and e a positive integer. Then for each i there is an isomorphism of eS -modules*

$$\mathrm{Tor}_i^R(M, {}^eR) \otimes_{{}^eR} {}^eS \simeq \mathrm{Tor}_i^S(M \otimes_R S, {}^eS).$$

Proof. We have a commutative square of ring maps:

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ {}^eR & \longrightarrow & {}^eS \end{array}$$

Since S is flat over R , eS is flat over eR . The result now follows from Lemma 2.1. \square

Lemma 2.3. *Let (R, \mathfrak{m}) be a local ring of prime characteristic p which is F -finite. Let x be an indeterminate over R , $S := R[x]_{\mathfrak{m}R[x]}$, and $T := ({}^eR)[x]_{\mathfrak{n}({}^eR)[x]}$, where \mathfrak{n} is the maximal ideal of eR . Then*

- (a) eS is a free T -module of rank p^e .
- (b) T is a finitely generated S -module.
- (c) S is F -finite.
- (d) For each R -complex M with $H^*(M)$ bounded below and for each i , there is an isomorphism of eS -modules

$$\mathrm{Ext}_S^i({}^eS, M \otimes_R S) \simeq \mathrm{Hom}_T({}^eS, \mathrm{Ext}_R^i({}^eR, M) \otimes_R S).$$

Proof. Let $A = R[x]$, $B = ({}^eR)[x]$, and $C = ({}^eR)[x^{\frac{1}{p^e}}] \cong {}^eA$. Note that C is a free B -module of rank p^e and B is a f.g. A -module. Let $U = A \setminus \mathfrak{m}A$, $V = B \setminus \mathfrak{n}B$, and $W = C \setminus \mathfrak{n}C$. Then $A_U = S$. It is straightforward to check that $T = B_V = B_U$ and ${}^eS = C_W = C_V$. Hence, (a), (b), and (c) are immediate.

We have the following isomorphisms of eS -modules.

$$\begin{aligned} \mathrm{Hom}_T({}^eS, \mathrm{Ext}_R^i({}^eR, M) \otimes_R S) &\cong \mathrm{Hom}_T({}^eS, \mathrm{Ext}_S^i(T, M \otimes_R S)) \\ &\cong \mathrm{Hom}_T({}^eS, \mathrm{H}^i(\mathrm{RHom}_S(T, M \otimes_R S))) \\ &\cong \mathrm{H}^i(\mathrm{RHom}_T({}^eS, \mathrm{RHom}_S(T, M \otimes_R S))) \\ &\cong \mathrm{H}^i(\mathrm{RHom}_S({}^eS, M \otimes_R S)) \\ &\cong \mathrm{Ext}_S^i({}^eS, M \otimes_R S). \end{aligned}$$

The first isomorphism follows since S is flat over R , eR is finitely generated over R , and $\mathrm{H}^*(M)$ is bounded below (see [3, Lemma 4.4(F)]). The third isomorphism holds as eS is a free T -module. \square

Corollary 2.4. *With the notation as in part (d) of Lemma 2.3, for each i we have that $\mathrm{Ext}_R^i({}^eR, M) = 0$ if and only if $\mathrm{Ext}_S^i({}^eS, M \otimes_R S) = 0$.*

Proof. Since $\mathrm{Hom}_T({}^eS, -)$ and $- \otimes_R S$ are faithful functors, the result follows from part (4) of Lemma 2.3. \square

The following result is also well-known:

Lemma 2.5. *Let R be a commutative Noetherian ring, M, N R -complexes, and I an injective R -module.*

(a) *For all i we have isomorphisms*

$$\mathrm{Hom}_R(\mathrm{Tor}_i^R(M, N), I) \cong \mathrm{Ext}_R^i(M, \mathrm{Hom}_R(N, I)).$$

(b) *Suppose $\mathrm{H}_*(M)$ is bounded below, $\mathrm{H}_i(M)$ is finitely generated for all i , and $\mathrm{H}^*(N)$ is bounded below. Then for all i we have isomorphisms*

$$\mathrm{Tor}_i^R(M, \mathrm{Hom}_R(N, I)) \cong \mathrm{Hom}_R(\mathrm{Ext}_R^i(M, N), I).$$

Proof. Using adjunction and [3, Lemma 4.4(I)], we have the following isomorphisms in $\mathrm{D}(R)$:

$$\begin{aligned} \mathrm{Hom}_R(M \otimes_R^{\mathbf{L}} N, I) &\simeq \mathrm{RHom}_R(M \otimes_R^{\mathbf{L}} N, I) \\ &\simeq \mathrm{RHom}_R(M, \mathrm{RHom}_R(N, I)) \\ &\simeq \mathrm{RHom}_R(M, \mathrm{Hom}_R(N, I)) \end{aligned}$$

and

$$M \otimes_R^{\mathbf{L}} \mathrm{Hom}_R(N, I) \simeq \mathrm{Hom}_R(\mathrm{RHom}_R(M, N), I).$$

Taking homology and using that $\mathrm{Hom}_R(-, I)$ is an exact functor yields the desired isomorphisms. \square

For an R -complex M , let M^\sharp denote the complex which has the same underlying graded module as M and whose differentials are all zero. Let $\mathrm{fd}_R M$ denote the flat dimension of M ; that is,

$$\mathrm{fd}_R M = \inf\{\sup \mathrm{H}_*(F^\sharp) \mid F \simeq M, F \text{ semi-flat}\}.$$

Similarly, $\mathrm{id}_R M$ will denote the injective dimension of M , i.e.,

$$\mathrm{id}_R M = \inf\{\sup \mathrm{H}^*(I^\sharp) \mid I \simeq M, I \text{ semi-injective}\}.$$

Corollary 2.6. *Let (R, \mathfrak{m}) be a local ring, $E = E_R(R/\mathfrak{m})$, and let $(-)^v$ denote the functor $\text{Hom}_R(-, E)$. Let M be an R -complex. Then*

- (a) $\text{fd}_R M \leq \text{id}_R M^v$ with equality if $H_*(M)$ is bounded below.
- (b) If $H^*(M)$ is bounded below, then $\text{id}_R M = \text{fd}_R M^v$

Proof. Using [3, Proposition 5.3.F] and Lemma 2.5 with $I = E$, we have:

$$\begin{aligned} \text{fd}_R M &= \sup\{j \mid \text{Tor}_j^R(R/\mathfrak{p}, M) \neq 0 \text{ for some } \mathfrak{p} \in \text{Spec } R\} \\ &= \sup\{j \mid \text{Tor}_j^R(R/\mathfrak{p}, M)^v \neq 0 \text{ for some } \mathfrak{p} \in \text{Spec } R\} \\ &= \sup\{j \mid \text{Ext}_R^j(R/\mathfrak{p}, M^v) \neq 0 \text{ for some } \mathfrak{p} \in \text{Spec } R\} \\ &\leq \text{id}_R M^v, \end{aligned}$$

where equality holds in the last line if $H^*(M^v)$ is bounded below, or equivalently, if $H_*(M)$ is bounded below. Part (b) is proved similarly. □

We note the following remark, which will be needed in the subsequent sections:

Remark 2.7. Let S be a faithfully flat R -algebra and M an R -complex. Then

- (a) $\text{fd}_R M = \text{fd}_S M \otimes_R S$;
- (b) If $H^*(M)$ is bounded below, then $\text{id}_R M \leq \text{id}_S M \otimes_R S$.

Proof. For part (a), note that $\text{fd}_R M \geq \text{fd}_S M \otimes_R S$, since $-\otimes_R S$ preserves quasi-isomorphisms and $F \otimes_R S$ is a semi-flat S -complex whenever F is a semi-flat R -complex. For the reverse inequality, we have by [3, Proposition 5.3.F],

$$\begin{aligned} \text{fd}_R M &= \sup\{j \mid \text{Tor}_j^R(R/\mathfrak{p}, M) \neq 0 \text{ for some } \mathfrak{p} \in \text{Spec } R\} \\ &= \sup\{j \mid \text{Tor}_j^R(R/\mathfrak{p}, M) \otimes_R S \neq 0 \text{ for some } \mathfrak{p} \in \text{Spec } R\} \\ &= \sup\{j \mid \text{Tor}_j^S(S/\mathfrak{p}S, M \otimes_R S) \neq 0 \text{ for some } \mathfrak{p} \in \text{Spec } R\} \\ &\leq \text{fd}_S M \otimes_R S. \end{aligned}$$

For part (b), we have by [3, Proposition 5.3.I] that

$$\begin{aligned} \text{id}_R M &= \sup\{j \mid \text{Ext}_R^j(R/\mathfrak{p}, M) \neq 0 \text{ for some } \mathfrak{p} \in \text{Spec } R\} \\ &= \sup\{j \mid \text{Ext}_R^j(R/\mathfrak{p}, M) \otimes_R S \neq 0 \text{ for some } \mathfrak{p} \in \text{Spec } R\} \\ &= \sup\{j \mid \text{Ext}_S^j(S/\mathfrak{p}S, M \otimes_R S) \neq 0 \text{ for some } \mathfrak{p} \in \text{Spec } R\} \\ &\leq \text{id}_S M \otimes_R S. \end{aligned}$$

□

Finally, we will need the following result for zero-dimensional rings. It is a special case of Theorem 1.1 of [6] (or more properly, its dual), but as the proof is short, we include it here for the reader's convenience:

Proposition 2.8. *Let (R, \mathfrak{m}, k) be a zero-dimensional local ring of prime characteristic p . Let M be an R -module and $e \geq \log_p \lambda(R)$ an integer, where $\lambda(-)$ denotes length. If $\text{Ext}_R^i(eR, M) = 0$ for some $i > 0$ then M is injective.*

Proof. By [4, Proposition 4.1 and Corollary 5.3], if M has finite injective dimension then $\text{id}_R M \leq \dim R$. Hence, it suffices to show $\text{id}_R M < \infty$. By replacing M with a syzygy of an injective resolution of M , we may assume $\text{Ext}_R^1(eR, M) = 0$. Since

$p^e \geq \lambda(R)$, we have $\mathfrak{m}^{p^e} = 0$. Then $\mathfrak{m} \cdot {}^eR = 0$ and thus eR is a k -vector space. Hence, ${}^eR \cong k^\ell$ as R -modules, for some (possibly infinite) $\ell > 0$. Thus, the condition $\text{Ext}_R^1({}^eR, M) = 0$ implies $\text{Ext}_R^1(k, M) = 0$. Hence, M is injective. \square

3. THE COHEN-MACAULAY CASE

We first prove the following elementary result concerning maps to injective modules.

Lemma 3.1. *Let R be a commutative ring and T a finitely generated faithful R -module. Suppose $\alpha : M \rightarrow I$ is a homomorphism of R -modules where I is an injective module. If $\alpha_* : \text{Hom}_R(T, M) \rightarrow \text{Hom}_R(T, I)$ is surjective, then so is α .*

Proof. Let $e \in I$ and $f : R \rightarrow I$ be defined by $f(r) = re$. As T is a finitely generated faithful module, there exists an R -monomorphism $g : R \rightarrow T^n$ for some n . Let $g(1) = (u_1, \dots, u_n)$. As I is injective, there exists an R -homomorphism $h : T^n \rightarrow I$ such that $f = hg$. As α_* is surjective, so is $(\alpha_*)^n : \text{Hom}_R(T^n, M) \rightarrow \text{Hom}_R(T^n, I)$. Thus, there exists $w_1, \dots, w_n \in \text{Hom}_R(T, M)$ such that $h = (\alpha w_1, \dots, \alpha w_n)$. Then $e = h(g(1)) = h(u_1, \dots, u_n) = \alpha(w_1(u_1)) + \dots + \alpha(w_n(u_n)) \in \text{im } \alpha$. As e is arbitrary, α is surjective. \square

For an R -complex M and $\mathfrak{p} \in \text{Spec } R$, we let $\mu_R^i(\mathfrak{p}, M) := \dim_{k(\mathfrak{p})} \text{Ext}_{R_{\mathfrak{p}}}^i(k(\mathfrak{p}), M_{\mathfrak{p}})$. If $H^*(M)$ is bounded below, $\mu_i(\mathfrak{p}, M)$ is the number (possibly infinite) of copies of $E_R(R/\mathfrak{p})$ in I^i , where I is a minimal semi-injective resolution of M .

Theorem 3.2. *Let (R, \mathfrak{m}, k) be a d -dimensional Cohen-Macaulay local ring of prime characteristic p and which is F -finite. Let $e \geq \log_p e(R)$ be an integer and M an R -complex such that $H^*(M)$ is bounded above. Suppose there exists an integer $t > \sup H^*(M)$ such that $\text{Ext}_R^i({}^eR, M) = 0$ for $t \leq i \leq t + r - 1$, where $r = \max\{1, d\}$. Then M has finite injective dimension.*

Proof. If $\text{id}_R M < t - 1$ there is nothing to prove. Otherwise, let J be a minimal semi-injective resolution of M and $Z := Z^{t-1}(J)$ be the (necessarily nonzero) subcomplex consisting of the cycles of degree $t - 1$ of J . As $t - 1 \geq \sup H^*(M)$, $J^{\geq t-1}$ is a minimal semi-injective resolution of Z and $\text{id}_R M = \text{id}_R Z$. Furthermore, from the exact sequence of complexes

$$0 \rightarrow J^{\geq t-1} \rightarrow J \rightarrow J^{< t-1} \rightarrow 0$$

we have that $\text{Ext}_R^i({}^eR, Z) \cong \text{Ext}_R^i({}^eR, M)$ for all $i \geq t$. Hence, without loss of generality, we may assume (after shifting) that M is a module concentrated in degree zero and $\text{Ext}_R^i({}^eR, M) = 0$ for $i = 1, \dots, r$. Also, by replacing R with $R[x]_{\mathfrak{m}R[x]}$, if necessary, we may assume R has an infinite residue field (Lemma 2.3 and Remark 2.7).

We proceed by induction on d , with the case $d = 0$ being established by Proposition 2.8. Suppose $d \geq 1$ (so $r = d$) and let $\mathfrak{p} \neq \mathfrak{m}$ be a prime ideal of R . As R is F -finite, we have $\text{Ext}_{R_{\mathfrak{p}}}^i({}^eR_{\mathfrak{p}}, M_{\mathfrak{p}}) = 0$ for $1 \leq i \leq d$. As $d \geq \max\{1, \dim R_{\mathfrak{p}}\}$ and $e(R) \geq e(R_{\mathfrak{p}})$ (see [12]), we have $\text{id}_R M_{\mathfrak{p}} < \infty$ by the induction hypothesis. Hence, $\text{id}_R M_{\mathfrak{p}} \leq \dim R_{\mathfrak{p}} \leq d - 1$ by [4, Proposition 4.1 and Corollary 5.3]. It follows that $\mu_i(\mathfrak{p}, M) = 0$ for all $i \geq d$ and all $\mathfrak{p} \neq \mathfrak{m}$.

For convenience, we let S denote the R -algebra eR and \mathfrak{n} the maximal ideal of S . As S/\mathfrak{n} is infinite, we may choose a system of parameters $\mathbf{x} = x_1, \dots, x_d \in \mathfrak{n}$

such that (\mathbf{x}) is a minimal reduction of \mathfrak{n} . Then $\lambda_S(S/(\mathbf{x})) = e(S) = e(R)$ and $\mathfrak{m} \cdot S/(\mathbf{x}) = \mathfrak{n}^{[p^e]}S/(\mathbf{x}) = 0$, as $p^e \geq \lambda_S(S/(\mathbf{x}))$.

As J is a minimal injective resolution of M , we have by assumption that

$$(3.1) \quad \mathrm{Hom}_R(S, J^0) \xrightarrow{\phi^0} \mathrm{Hom}_R(S, J^1) \rightarrow \cdots \rightarrow \mathrm{Hom}_R(S, J^d) \xrightarrow{\phi^d} \mathrm{Hom}_R(S, J^{d+1})$$

is exact. Let L be the injective S -envelope of $\mathrm{coker} \phi^d$ and $\psi : \mathrm{Hom}_R(S, J^{d+1}) \rightarrow L$ the induced map. Hence,

$$0 \rightarrow \mathrm{Hom}_R(S, J^0) \rightarrow \cdots \xrightarrow{\phi^d} \mathrm{Hom}_R(S, J^{d+1}) \xrightarrow{\psi} L$$

is acyclic and in fact the start of an injective S -resolution of $\mathrm{Hom}_R(S, M)$. Setting $\bar{S} = S/(\mathbf{x})$ and applying $\mathrm{Hom}_S(\bar{S}, -)$ to the above resolution yields an exact sequence

$$(3.2) \quad \mathrm{Hom}_S(\bar{S}, \mathrm{Hom}_R(S, J^d)) \xrightarrow{\bar{\phi}^d} \mathrm{Hom}_S(\bar{S}, \mathrm{Hom}_R(S, J^{d+1})) \xrightarrow{\bar{\psi}} \mathrm{Hom}_S(\bar{S}, L).$$

The exactness holds as $\mathrm{pd}_S \bar{S} = d$ and thus $\mathrm{Ext}_S^{d+1}(\bar{S}, \mathrm{Hom}_S(S, M)) = 0$.

Since \bar{S} is a finitely generated R -module and annihilated by \mathfrak{m} , we have $\bar{S} \cong k^t$ as R -modules for some t . Thus, the exact sequence (3.2) is naturally isomorphic to

$$\mathrm{Hom}_R(k^t, J^d) \xrightarrow{\bar{\phi}^d} \mathrm{Hom}_R(k^t, J^{d+1}) \xrightarrow{\bar{\psi}} \mathrm{Hom}_S(\bar{S}, L).$$

As J is minimal, we have $\bar{\phi}^d$ is the zero map and hence $\bar{\psi}$ is injective.

Claim: ψ is injective.

Proof: Let $K = \ker \psi$. Applying $\mathrm{Hom}_S(\bar{S}, -)$ to

$$0 \rightarrow K \rightarrow \mathrm{Hom}_R(S, J^{d+1}) \xrightarrow{\psi} L$$

we see that $\mathrm{Hom}_S(\bar{S}, K) = 0$. Since $\mu_{d+1}(\mathfrak{p}, M) = 0$ for all primes $\mathfrak{p} \neq \mathfrak{m}$, we obtain that $J^{d+1} = \bigoplus_{\alpha \in I} E_R(k)$ for some (possibly infinite) index set I . Since S is a finite R -module, we have $\mathrm{Hom}_R(S, J^{d+1})_{\mathfrak{p}} \cong \mathrm{Hom}_{R_{\mathfrak{p}}}(S_{\mathfrak{p}}, J_{\mathfrak{p}}^{d+1}) = 0$ for all $\mathfrak{p} \neq \mathfrak{m}$. Hence, $\mathrm{Hom}_R(S, J^{d+1})_{\mathfrak{q}} = 0$ for all $\mathfrak{q} \in \mathrm{Spec} S$, $\mathfrak{q} \neq \mathfrak{n}$. Thus $\mathrm{Hom}_R(S, J^{d+1})$, and consequently K , is \mathfrak{n} -torsion. Thus, if $K \neq 0$, we must have $\mathrm{Hom}_S(\bar{S}, K) \neq 0$. We conclude $K = 0$ and ψ is injective.

As ψ is injective we have from (3.1) that $\phi^d = 0$ and hence ϕ^{d-1} is surjective. Since S is a finitely generated faithful R -module, we see by Lemma 3.1 that the map $J^{d-1} \rightarrow J^d$ is surjective. As J is minimal injective resolution of M , this implies $J^i = 0$ for all $i > d$ and so $\mathrm{id}_R M < \infty$. □

As a corollary, we obtain the equivalence of conditions (a) and (c) of Theorem 1.1:

Corollary 3.3. *Let (R, \mathfrak{m}) be a d -dimensional Cohen-Macaulay local ring of prime characteristic p and M an R -complex such that $H_*(M)$ is bounded above. Suppose there exist integers $e \geq \log_p e(R)$ and $t > \sup H_*(M)$ such that $\mathrm{Tor}_i^R({}^e R, M) = 0$ for $t \leq i \leq t + r - 1$, where $r = \max\{1, d\}$. Then M has finite flat dimension.*

Proof. By [11, Section 3] there exists a faithfully flat extension S of R such that S is a d -dimensional CM local ring with an algebraically closed residue field and $e(S) = e(R)$. Furthermore, by Corollary 2.2, $\mathrm{Tor}_i^S({}^e S, M \otimes_R S) = 0$ for $t \leq i \leq t + r - 1$. Hence, by replacing R with S and M with $M \otimes_R S$, we may assume R is F -finite. By Lemma 2.5(a), $\mathrm{Ext}_R^i({}^e R, M^\vee) \cong \mathrm{Tor}_i^R({}^e R, M)^\vee = 0$ for $t \leq i \leq t + r - 1$.

As $\sup H^*(M^\vee) = \sup H_*(M)$, we have by Theorem 3.2 that $\text{id}_R M^\vee < \infty$. Hence, $\text{fd}_R M < \infty$ by Lemma 2.6. \square

4. THE GENERAL CASE

We begin this section by proving a basic result concerning $E = E_R(k)$, the injective hull of the residue field of a local ring (R, \mathfrak{m}, k) .

Lemma 4.1. *Let (R, \mathfrak{m}, k) be a local ring. Then*

$$(0 :_E (0 :_R \mathfrak{m})) = \mathfrak{m}E.$$

Proof. One containment is clear. For the reverse inclusion, since $E \cong E_{\hat{R}}(\hat{R}/\hat{\mathfrak{m}})$, $\hat{\mathfrak{m}}E = \mathfrak{m}E$ and $(0 :_{\hat{R}} \hat{\mathfrak{m}}) = (0 :_R \mathfrak{m})\hat{R}$, we may replace R by \hat{R} and assume R is complete. Consider the composition of maps

$$(4.1) \quad \text{Hom}_R(R/(0 :_R \mathfrak{m}), E) \cong (0 :_E (0 :_R \mathfrak{m})) \rightarrow E \rightarrow E/\mathfrak{m}E \cong E \otimes_R R/\mathfrak{m}.$$

Dualizing, we have the composition

$$(E \otimes_R R/\mathfrak{m})^\vee \cong (0 :_R \mathfrak{m}) \rightarrow R \rightarrow R/(0 :_R \mathfrak{m}),$$

which is clearly the zero map. Thus, the composition (4.1) is the zero map as well, implying $(0 :_E (0 :_R \mathfrak{m})) \subseteq \mathfrak{m}E$. \square

We use the above lemma to prove the following:

Lemma 4.2. *Let (R, \mathfrak{m}, k) be a local ring and $\phi : J \rightarrow J'$ a homomorphism of injective R -modules. Suppose $\text{Hom}_R(R/\mathfrak{m}, J) \xrightarrow{\phi_*} \text{Hom}_R(R/\mathfrak{m}, J')$ is zero. Then $\phi(J) \subseteq \mathfrak{m}J'$.*

Proof. It suffices to prove the lemma in the case $J = E_R(R/\mathfrak{p})$ and $J' = E_R(R/\mathfrak{q})$, for $\mathfrak{p}, \mathfrak{q} \in \text{Spec } R$,

Case 1: $\mathfrak{q} \neq \mathfrak{m}$.

Then $\mathfrak{m}J' = \mathfrak{m}E_R(R/\mathfrak{q}) = \mathfrak{m}R_{\mathfrak{q}} \cdot E_R(R/\mathfrak{q}) = J'$, as $E_R(R/\mathfrak{q})$ is an $R_{\mathfrak{q}}$ -module. So the lemma holds trivially.

Case 2: $\mathfrak{q} = \mathfrak{m}$ and $\mathfrak{p} \neq \mathfrak{m}$.

Since $J = E_R(R/\mathfrak{p})$ is an $R_{\mathfrak{p}}$ -module, we have

$$(0 :_R \mathfrak{m})\phi(J) = \phi((0 :_R \mathfrak{m})R_{\mathfrak{p}} \cdot J) = \phi(0) = 0.$$

Hence, $\phi(J) \subseteq (0 :_{J'} (0 :_R \mathfrak{m})) = \mathfrak{m}J'$ by Lemma 4.1.

Case 3: $\mathfrak{p} = \mathfrak{q} = \mathfrak{m}$.

In this case, ϕ is multiplication by some element $s \in \hat{R}$. If $s \notin \hat{\mathfrak{m}}$, then ϕ is an isomorphism, contradicting that $\text{Hom}_R(R/\mathfrak{m}, \phi)$ is the zero map. Thus, $s \in \hat{\mathfrak{m}}$. Hence, $\phi(J) \subseteq \hat{\mathfrak{m}}J' = \mathfrak{m}J'$. \square

Lemma 4.3. *Let (R, \mathfrak{m}) be a local ring of depth zero and let ℓ be an integer such that $(0 :_R \mathfrak{m}) \not\subseteq \mathfrak{m}^\ell$. Let J be an injective module such that $\mu(\mathfrak{m}, J) \neq 0$. Then $(0 :_J \mathfrak{m}^\ell) \not\subseteq \mathfrak{m}J$.*

Proof. It suffices to consider the case $J = E := E_R(k)$. Since the composition $(0 :_R \mathfrak{m}) \rightarrow R \rightarrow R/\mathfrak{m}^\ell$ is nonzero, the composition

$$(R/\mathfrak{m}^\ell)^\vee \cong (0 :_E \mathfrak{m}^\ell) \rightarrow E \rightarrow E/\mathfrak{m}E \cong \text{Hom}_R(R/\mathfrak{m}, R)^\vee$$

is also nonzero. Hence, $(0 :_E \mathfrak{m}^\ell) \not\subseteq \mathfrak{m}E$. \square

Lemma 4.4. *Let $\phi : (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ be a local homomorphism such that S is a finitely generated R -module and $\text{depth } S = 0$. Let ℓ be an integer such that $(0 :_S \mathfrak{n}) \not\subseteq \mathfrak{n}^\ell$ and suppose $\mathfrak{m}S \subseteq \mathfrak{n}^\ell$. Let $J^1 \xrightarrow{\sigma} J^2 \xrightarrow{\tau} J^3$ be a sequence of maps of injective modules such that $\text{Hom}_R(R/\mathfrak{m}, \sigma) = \text{Hom}_R(R/\mathfrak{m}, \tau) = 0$. If $\text{Hom}_R(S, J^1) \xrightarrow{\sigma_*} \text{Hom}_R(S, J^2) \xrightarrow{\tau_*} \text{Hom}_R(S, J^3)$ is exact then $\mu(\mathfrak{m}, J^2) = 0$.*

Proof. Let $\widetilde{J}^i = \widetilde{\text{Hom}}_R(S, J^i)$ for $i = 1, 2, 3$, which are injective S -modules. Since $\mathfrak{m}S \subseteq \mathfrak{n}^\ell$, we have that $S/\mathfrak{n}^\ell \cong k^r$ as R -modules for some $r > 0$, where $k = R/\mathfrak{m}$. From the commutative diagram

$$\begin{array}{ccc} \text{Hom}_S(S/\mathfrak{n}^\ell, \widetilde{J}^1) & \xrightarrow{\overline{\sigma}_*} & \text{Hom}_S(S/\mathfrak{n}^\ell, \widetilde{J}^2) \\ \downarrow \cong & & \downarrow \cong \\ \text{Hom}_R(S/\mathfrak{n}^\ell, J^1) & \longrightarrow & \text{Hom}_R(S/\mathfrak{n}^\ell, J^2) \\ \downarrow \cong & & \downarrow \cong \\ \oplus \text{Hom}_R(k, J^1) & \xrightarrow{\oplus \overline{\sigma}} & \oplus \text{Hom}_R(k, J^2) \end{array}$$

As $\overline{\sigma}$ is the zero map by hypothesis, we see that $\overline{\sigma}_*$ is zero. Similarly, the map

$$\overline{\tau}_* : \text{Hom}_S(S/\mathfrak{n}^\ell, \widetilde{J}^2) \rightarrow \text{Hom}_S(S/\mathfrak{n}^\ell, \widetilde{J}^3)$$

is zero. This implies that $(0 :_{\widetilde{J}^2} \mathfrak{n}^\ell) \subseteq \ker \overline{\tau}_*$. As $\overline{\sigma}_*$ is zero, we also have that the map $\text{Hom}_S(S/\mathfrak{n}, \widetilde{J}^1) \rightarrow \text{Hom}_S(S/\mathfrak{n}, \widetilde{J}^2)$ is zero. By Lemma 4.2, this implies that $\text{im } \sigma_* \subseteq \mathfrak{n}J^2$.

Suppose $\mu(\mathfrak{m}, J^2) \neq 0$. Since $\text{Hom}_S(S, E_R(R/\mathfrak{m})) \cong E_S(S/\mathfrak{n})$ by [13, Lemma 3.7], we then have $\mu(\mathfrak{n}, \widetilde{J}^2) \neq 0$. By Lemma 4.3, we have that $(0 :_{\widetilde{J}^2} \mathfrak{n}^\ell) \not\subseteq \mathfrak{n}\widetilde{J}^2$. Hence, $\ker \tau_* \not\subseteq \text{im } \sigma_*$, a contradiction. Therefore, $\mu(\mathfrak{m}, J^2) = 0$. \square

Theorem 4.5. *Let (R, \mathfrak{m}, k) be a d -dimensional local ring of prime characteristic p which is F -finite. Let M be an R -complex such that $\text{H}^*(M)$ is bounded above. Suppose there exists an integer $t > \sup \text{H}^*(M)$ such that for infinitely many integers e one has $\text{Ext}_R^i({}^e R, M) = 0$ for $t \leq i \leq t + r - 1$, where $r = \max\{1, d\}$. Then M has finite injective dimension.*

Proof. Precisely as in the initial paragraph of the proof of Theorem 3.2, we may assume M is a module concentrated in degree 0 and $t = 1$. We proceed by induction on d , with the case $d = 0$ being covered by Proposition 2.8. Suppose now that $d \geq 1$ (so $r = d$) and let $\mathfrak{p} \neq \mathfrak{m}$ be a prime ideal. Since R is F -finite, $\text{Ext}_{R_{\mathfrak{p}}}^i({}^e R_{\mathfrak{p}}, M_{\mathfrak{p}}) = \text{Ext}_R^i({}^e R, M)_{\mathfrak{p}} = 0$ for infinitely many e and $i = 1, \dots, d$. Since $d \geq \max\{1, \dim R_{\mathfrak{p}}\}$, we have $\text{id}_R M_{\mathfrak{p}} < \infty$ by the induction hypothesis. Hence, $\text{id}_R M_{\mathfrak{p}} \leq \dim R_{\mathfrak{p}} \leq d - 1$. Thus, $\mu_i(\mathfrak{p}, M) = 0$ for all $i \geq d$ and all $\mathfrak{p} \neq \mathfrak{m}$. It suffices to prove $\mu_d(\mathfrak{m}, M) = 0$.

If R is Cohen-Macaulay we are done by Theorem 3.2. Hence we may assume $s := \text{depth } R < d$. Let $e \geq 1$ be arbitrary and let T denote the local ring ${}^e R$ and

\mathfrak{q} the maximal ideal of T . Let $\mathbf{x} = x_1, \dots, x_s \in \mathfrak{q}$ be a maximal regular sequence in T and set $S := T/(\mathbf{x})$ and $\mathfrak{n} := \mathfrak{q}S$. Since $\text{depth } S = 0$, there exists an integer ℓ (independent of e) such that $(0 :_S \mathfrak{n}) \not\subseteq \mathfrak{n}^\ell$. Now choose e sufficiently large such that $p^e \geq \ell$ and $\text{Ext}_R^i(T, M) = 0$ for $i = 1, \dots, d$. Let

$$J := 0 \rightarrow J^0 \rightarrow J^1 \rightarrow J^2 \rightarrow \dots$$

be a minimal injective resolution of M , and for each i let \widetilde{J}^i denote $\text{Hom}_R(T, J^i)$. As $\text{Ext}_R^i(T, M) = 0$ for $1 \leq i \leq d$, we see that

$$0 \rightarrow \widetilde{J}^0 \rightarrow \widetilde{J}^1 \rightarrow \dots \rightarrow \widetilde{J}^d \rightarrow \widetilde{J}^{d+1}$$

is part of an injective T -resolution of $\widetilde{M} := \text{Hom}_R(T, M)$. Since $\text{pd}_T S = s$ we have that $\text{Ext}_T^i(S, \widetilde{M}) = 0$ for $i > s$. In particular, as $d > s$ and $\text{Hom}_T(S, \widetilde{J}^i) \cong \text{Hom}_R(S, J^i)$ for all i , we have that

$$\text{Hom}_R(S, J^{d-1}) \rightarrow \text{Hom}_R(S, J^d) \rightarrow \text{Hom}_R(S, J^{d+1})$$

is exact. Since $\mathfrak{m}S \subseteq \mathfrak{n}^{\lfloor p^e \rfloor} \subseteq \mathfrak{n}^\ell$, we obtain that $\mu_d(\mathfrak{m}, M) = \mu(\mathfrak{m}, J^d) = 0$ by Lemma 4.4. Thus, $\text{id}_R M < \infty$. \square

We now obtain the equivalence of conditions (a) and (b) of Theorem 1.1:

Corollary 4.6. *Let (R, \mathfrak{m}) be a d -dimensional local ring of prime characteristic p and M an R -complex such that $\text{H}_*(M)$ is bounded above. Suppose there exist an integer $t > \sup \text{H}_*(M)$ such that for infinitely many integers e one has $\text{Tor}_i^R({}^e R, M) = 0$ for $t \leq i \leq t + r - 1$, where $r = \max\{1, d\}$. Then M has finite flat dimension.*

Proof. The argument is similar to the proof of Corollary 3.3, except one uses Theorem 4.5 in place of Theorem 3.2. \square

5. THE CASE OF COMPLETE INTERSECTIONS

In this section, we prove that a theorem of Avramov and Miller [2] concerning finitely generated modules over complete intersections holds for arbitrary modules, and in fact any complex whose homology is bounded above. The proof mostly follows the argument of Dutta [7], until the end when we apply [6, Theorem 1.1].

Theorem 5.1. *Let (R, \mathfrak{m}) be a local complete intersection ring of prime characteristic p . Let M be an R -complex such that $\text{H}_*(M)$ is bounded above. Suppose $\text{Tor}_i^R({}^e R, M) = 0$ for some $e > 0$ and some $i > \sup \text{H}_*(M)$. Then M has finite flat dimension.*

Proof. Without loss of generality, we may assume R is complete. Then $R \cong A/(\mathbf{x})$ where (A, \mathfrak{n}) is a complete regular local ring and $\mathbf{x} = x_1, \dots, x_r \in \mathfrak{n}^2$ is a regular sequence. Since A is regular the e th iteration of the Frobenius map $f^e : A \rightarrow A$ is flat. Thus, the map $h : A/(\mathbf{x}) \rightarrow A/(\mathbf{x}^{\mathfrak{P}^e})$, given by $h(\bar{r}) = \overline{r^{\mathfrak{P}^e}}$ for $\bar{r} \in A/(\mathbf{x})$, is flat as well. Let T denote the ring $A/(\mathbf{x}^{\mathfrak{P}^e})$ viewed as an R -algebra via h . Also, let $S := T/(\mathbf{x})$ and $\widetilde{M} := T \otimes_R M$. Note that as an R -algebra, $S \cong {}^e R$. Since T is flat over R , we have isomorphisms in $\text{D}(R)$

$$S \otimes_R^{\mathbf{L}} M \simeq (S \otimes_T^{\mathbf{L}} T) \otimes_R^{\mathbf{L}} M \simeq S \otimes_T^{\mathbf{L}} (T \otimes_R^{\mathbf{L}} M) \simeq S \otimes_T^{\mathbf{L}} \widetilde{M}.$$

Taking homology, we have an isomorphism for all j

$$(5.1) \quad \text{Tor}_j^R(S, M) \cong \text{Tor}_j^T(S, \widetilde{M}).$$

Claim 1: $\mathrm{Tor}_j^R({}^eR, M) = 0$ for all $j \geq i$.

Proof: It suffices to show that $\mathrm{Tor}_{i+1}^R(S, M) = 0$. By (5.1), it suffices to prove that whenever $\mathrm{Tor}_i^T(S, \widetilde{M}) = 0$ for some $i > \sup H_*(\widetilde{M}) = \sup H_*(M)$ then $\mathrm{Tor}_{i+1}^T(S, \widetilde{M}) = 0$. Let $\phi : T \rightarrow S$ be the canonical surjection and $K = \ker \phi$. As \mathbf{x} is a regular sequence on A , K has a finite filtration of T -submodules such that each factor module is isomorphic to S . Thus, $\mathrm{Tor}_i^T(K, \widetilde{M}) = 0$. As $\mathrm{Tor}_j^T(T, \widetilde{M}) = 0$ for all $j > \sup H_*(\widetilde{M})$, we obtain that $\mathrm{Tor}_{i+1}^T(S, \widetilde{M}) = 0$.

Claim 2: $\mathrm{Tor}_j^R({}^{e+1}R, M) = 0$ for all $j \geq i$.

Proof: As $f : A \rightarrow A$ is flat, by base change we have the induced map $g : T \rightarrow A/(\mathbf{x}^{\mathbf{p}^{e+1}})$ is also flat. Let T' denote the ring $A/(\mathbf{x}^{\mathbf{p}^{e+1}})$ viewed as a T -algebra via g (and hence as an R -algebra via gh). Note that $T'/(\mathbf{x}) = {}^{e+1}R$. Let $\widehat{M} = T' \otimes_R M = T' \otimes_T \widetilde{M}$. Since T' is flat over R , we have isomorphisms in $\mathrm{D}(R)$

$$T'/(\mathbf{x}) \otimes_R^{\mathbf{L}} M \simeq (T'/(\mathbf{x}) \otimes_{T'}^{\mathbf{L}} T') \otimes_R^{\mathbf{L}} M \simeq T'/(\mathbf{x}) \otimes_{T'}^{\mathbf{L}} (T' \otimes_R^{\mathbf{L}} M) \simeq T'/(\mathbf{x}) \otimes_{T'}^{\mathbf{L}} \widehat{M}.$$

Taking homology, we have an isomorphism for all j

$$(5.2) \quad \mathrm{Tor}_j^R(T'/(\mathbf{x}), M) \cong \mathrm{Tor}_j^{T'}(T'/(\mathbf{x}), \widehat{M}).$$

Hence, it suffices to show that $\mathrm{Tor}_j^{T'}(T'/(\mathbf{x}), \widehat{M}) = 0$ for all $j \geq i$. By (5.1) and Claim 1, we have that $\mathrm{Tor}_j^T(T'/(\mathbf{x}), \widetilde{M}) = 0$ for all $j \geq i$. Since T' is flat over T , we obtain that $\mathrm{Tor}_j^{T'}(T'/(\mathbf{x}), \widehat{M}) = 0$ for $j \geq i$.

As $\mathbf{x} = x_1, \dots, x_r$ is a regular sequence on A and $T' = A/(\mathbf{x}^{\mathbf{p}^{e+1}})$, we have exact sequences of T -modules

$$(5.3) \quad 0 \rightarrow T'/(x_1, x_2^p, \dots, x_r^p) \rightarrow T'/(x_1^{p+1}, x_2^p, \dots, x_r^p) \rightarrow T'/(x_1^p, \dots, x_r^p) \rightarrow 0$$

$$(5.4) \quad 0 \rightarrow T'/(x_1^p) \rightarrow T'/(x_1^{p+1}, x_2^p, \dots, x_r^p) \rightarrow T'/(x_1, x_2^p, \dots, x_r^p) \rightarrow 0,$$

where the initial maps in (5.3) and (5.4) are multiplication by x_1^p and x_1 , respectively. Using that $\mathrm{Tor}_j^{T'}(T'/(x_1^p), \widehat{M}) = 0$ for $j \geq i$ in conjunction with (5.3) and (5.4), we get an injection

$$\mathrm{Tor}_j^{T'}(T'/(x_1, x_2^p, \dots, x_r^p), \widehat{M}) \hookrightarrow \mathrm{Tor}_j^{T'}(T'/(x_1^p, x_2^p, \dots, x_r^p), \widehat{M})$$

for all $j \geq i$ which is induced by multiplication by x_1^p , which is the zero map. Hence $\mathrm{Tor}_j^{T'}(T'/(x_1, x_2^p, \dots, x_r^p), \widehat{M}) = 0$ for all $j \geq i$. Repeating this argument for each of x_2, \dots, x_r yields that $\mathrm{Tor}_j^{T'}(T'/(\mathbf{x}), \widehat{M}) = 0$ for all $j \geq i$, which completes the proof of Claim 2.

Combining Claim 1 and Claim 2 (and iterating), we have $\mathrm{Tor}_j^R({}^eR, M) = 0$ for all $j \geq i > \sup H_*(M)$ and infinitely many e . Thus, M has finite flat dimension by [6, Theorem 1.1]. \square

We now deduce the dual version of this result for complexes of finite injective dimension:

Corollary 5.2. *Let (R, \mathfrak{m}) be a local complete intersection ring of prime characteristic p , and assume R is F -finite. Let M be an R -complex such that $H^*(M)$ is bounded above. Suppose $\mathrm{Ext}_i^R({}^eR, M) = 0$ for some $e > 0$ and some $i > \sup H^*(M)$. Then M has finite injective dimension.*

Proof. By the argument in the initial paragraph of Theorem 3.2, we may assume M is a module concentrated in degree zero. As R is F -finite, we have by Lemma 2.5 that $\mathrm{Tor}_i^R({}^eR, M^\vee) = 0$ for some positive integers i and e , where $(-)^{\vee}$ denotes the functor $\mathrm{Hom}_R(-, E_R(R/\mathfrak{m}))$. By Theorem 5.1, we have $\mathrm{fd}_R M^\vee < \infty$. Hence, by Lemma 2.6, $\mathrm{id}_R M = \mathrm{fd}_R M^\vee < \infty$. \square

REFERENCES

1. M. Auslander and D. A. Buchsbaum, *Homological dimension in Noetherian rings. II*, Trans. Amer. Math. Soc. **88** (1958), no. 1, 194–206.
2. L. L. Avramov and C. Miller, *Frobenius powers of complete intersections*, Math. Res. Lett. **8** (2001), no 1-2, 225–232.
3. L. L. Avramov, H.-B. Foxby, *Homological dimensions of unbounded complexes*, J. Pure Appl. Alg. **71** (1991), 129–155.
4. H. Bass, *Injective dimension in Noetherian rings*, Trans. Amer. Math. Soc. **102** (1962), 19–29.
5. L. W. Christensen, S. B. Iyengar, T. Marley, *Rigidity of Ext and Tor with coefficients in residue fields of a commutative noetherian ring*, Proc. Edinburgh Math. Soc., to appear.
6. D. J. Dailey, S. B. Iyengar, T. Marley, *Detecting finite flat dimension of modules via iterates of the Frobenius endomorphism*, J. Commutative Alg., to appear.
7. S. P. Dutta, *On modules of finite projective dimension over complete intersections*, Proc. Amer. Math. Soc. **131** (2003) no. 1, 113–116.
8. J. Herzog, *Ringe der Charakteristik p und Frobeniusfunktoren*, Math. Z. **140** (1974), 67–78.
9. C. Jensen, *On the vanishing of $\varprojlim^{(i)}$* , J. Algebra **15** (1970), 151–166.
10. J. Koh, K. Lee, *Some restrictions on the maps in minimal resolutions*, J. Algebra **202** (1998), 671–689.
11. E. Kunz, *Characterizations of regular local rings for characteristic p* , Am. J. Math. **91** (1969), 772–784.
12. C. Lech, *Inequalities related to certain couples of local rings*, Acta Math. **112** ((1964), 69–89.
13. T. Marley *The Frobenius functor and injective modules*, Proc. Amer. Math. Soc. **142** (2014) no. 6, 1911–1923.
14. C. Miller *The Frobenius endomorphism and homological dimensions*, Commutative Algebra (Grenoble/Lyon, 2001), 207–234, Contemp. Math. **331**, Amer. Math. Soc., Providence, RI, 2003.
15. T. Marley, M. Webb, *The acyclicity of the Frobenius functor for modules of finite flat dimension*, J. Pure Appl. Algebra **220** (2016), no. 8, 2886–2896.
16. C. Peskine, L. Szpiro, *Dimension projective finie et cohomologie locale. Applications à la démonstration de conjectures de M. Auslander, H. Bass et A. Grothendieck*, Inst. Hautes Études Sci. Publ. Math. No. 42 (1973), 47–119.
17. M. Raynaud, L. Gruson, *Critères de platitude et de projectivité. Techniques de “platification” d’un module*, Invent. Math. **13** (1971), 1–89.

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