Matlis reflexivity and change of rings

Tom Marley

University of Nebraska

March 11, 2017

Matlis duality

This is joint work with Doug Dailey.

Let R be a semilocal ring with maximal ideals $m_1, \ldots, m_t$ and J the Jacobson radical of R. Let $E = E_R(R/m_1) \oplus \cdots \oplus E_R(R/m_t)$. Let $(-)^\vee$ denote the Matlis dual functor $\mathrm{Hom}_R(-, E)$.

Matlis duality

This is joint work with Doug Dailey.

Let R be a semilocal ring with maximal ideals $m_1, \ldots, m_t$ and J the Jacobson radical of R. Let $E = E_R(R/m_1) \oplus \cdots \oplus E_R(R/m_t)$. Let $(-)^\vee$ denote the Matlis dual functor $\mathrm{Hom}_R(-, E)$.

Theorem (Matlis, 1958)

Suppose R is J-adically complete. Then $(-)^\vee$ gives an anti-equivalence of the full subcategories of Noetherian and Artinian R-modules. In particular, for any Noetherian or Artinian R-module M, the natural (evaluation) map

$$\phi_M : M \to M^{\vee\vee} = \mathrm{Hom}_R(\mathrm{Hom}_R(M, E), E))$$

is an isomorphism.

Matlis reflexivity

Let R be a (commutative) Noetherian ring, Λ the set of maximal ideals of R, and

$$E_R = \bigoplus_{m \in \Lambda} E_R(R/m).$$

As before, we let $(-)^\vee$ denote the functor $\mathrm{Hom}_R(-, E_R)$.

Matlis reflexivity

Let R be a (commutative) Noetherian ring, Λ the set of maximal ideals of R, and

$$E_R = \bigoplus_{m \in \Lambda} E_R(R/m).$$

As before, we let $(-)^\vee$ denote the functor $\mathrm{Hom}_R(-, E_R)$.

An R-module M is called Matlis reflexive (or, in this talk, simply reflexive) if the natural map $M \to M^{\vee\vee}$ is an isomorphism.

Matlis reflexivity

Let R be a (commutative) Noetherian ring, Λ the set of maximal ideals of R, and

$$E_R = \bigoplus_{m \in \Lambda} E_R(R/m).$$

As before, we let $(-)^\vee$ denote the functor $\mathrm{Hom}_R(-, E_R)$.

An R-module M is called Matlis reflexive (or, in this talk, simply reflexive) if the natural map $M \to M^{\vee\vee}$ is an isomorphism.

An R-module M is said to be minimax if there exists a short exact sequence $0 \to X \to M \to Y \to 0$ where X is Noetherian and Y is Artinian.

Belshoff-Enochs-Garcia-Rozas Theorem

If R is a complete semilocal ring then any minimax module is reflexive, as seen by Matlis duality and the five-lemma.

Belshoff-Enochs-Garcia-Rozas Theorem

If R is a complete semilocal ring then any minimax module is reflexive, as seen by Matlis duality and the five-lemma.

In fact, a converse to this fact is also true:

Theorem (BEGR, 1999)

Let R be a Noetherian ring and M an R-module. Suppose M is reflexive. Then

1. $R/\operatorname{Ann}_R M$ is a complete semilocal ring, and
2. M is a minimax module.

Questions

Let S be a multiplicatively closed set of a commutative Noetherian ring R, and let M be an R_S-module.

Questions

Let S be a multiplicatively closed set of a commutative Noetherian ring R, and let M be an R_S-module.

1. If M is reflexive as an R-module, must it be reflexive as an R_S-module?
2. If M is reflexive as a R_S-module, must it be reflexive as an R-module?

Questions

Let S be a multiplicatively closed set of a commutative Noetherian ring R, and let M be an R_S-module.

1. If M is reflexive as an R-module, must it be reflexive as an R_S-module?
2. If M is reflexive as a R_S-module, must it be reflexive as an R-module?

Note: In general, it is not the case that $E_{R_S} = \mathrm{Hom}_R(R_S, E_R)$.

Questions

Let S be a multiplicatively closed set of a commutative Noetherian ring R, and let M be an R_S-module.

1. If M is reflexive as an R-module, must it be reflexive as an R_S-module?
2. If M is reflexive as a R_S-module, must it be reflexive as an R-module?

Note: In general, it is not the case that $E_{R_S} = \operatorname{Hom}_R(R_S, E_R)$.

Example: (Melkersson-Schenzel, 1995) Let R be a local domain and p a prime which is neither minimal nor maximal. Then $\operatorname{Hom}_R(R_p, E_R) \ncong E_{R_p}$.

Answers

It turns out that Question 1 has an affirmative answer:

Theorem 1

M is reflexive as an R-module $\implies$ M is reflexive as an R_S-module.

Answers

It turns out that Question 1 has an affirmative answer:

Theorem 1

M is reflexive as an R-module $\implies$ M is reflexive as an R_S-module.

But Question 2 has a negative answer in general:

Theorem-Example: Let R be a Noetherian local domain of dimension at least two. Let $Q = R_{(0)}$ be the field of fractions of R. Then Q is not a reflexive R-module. (Note that Q is always a reflexive $Q = R_{(0)}$-module.)

More answers

Theorem 2

Let (R, m) be a local ring and M a reflexive R_p-module for some prime p. Suppose p is not minimal over $\operatorname{Ann}_R M$. Then M is reflexive as an R-module.

More answers

Theorem 2

Let (R, m) be a local ring and M a reflexive R_p-module for some prime p. Suppose p is not minimal over $\operatorname{Ann}_R M$. Then M is reflexive as an R-module.

To prove this, we need a couple results:

More answers

Theorem 2

Let (R, m) be a local ring and M a reflexive R_p-module for some prime p. Suppose p is not minimal over $\operatorname{Ann}_R M$. Then M is reflexive as an R-module.

To prove this, we need a couple results:

Lemma (Kunz et al, 1967)

Let (R, m, k) be a complete local domain which is not a field and F the field of fractions of R. Let V be a DVR with field of fractions F. Then $R \subseteq V$.

More answers

Theorem 2

Let (R, m) be a local ring and M a reflexive R_p-module for some prime p. Suppose p is not minimal over $\operatorname{Ann}_R M$. Then M is reflexive as an R-module.

To prove this, we need a couple results:

Lemma (Kunz et al, 1967)

Let (R, m, k) be a complete local domain which is not a field and F the field of fractions of R. Let V be a DVR with field of fractions F. Then $R \subseteq V$.

Proof: Choose $a \in m$ and let n be any integer relatively prime to char k. Consider $f(x) = x^n - (1 + a)$. By Hensel's Lemma, there exists $b \in R$ such that $f(b) = 0$.

Proof (continued)

So $b^n = 1 + a$.

Let v be the valuation associated to V. Then $nv(b) = v(1 + a)$. If $v(a) < 0$ then $v(1 + a) < 0$. Hence, $v(b) \leq -1$ and $v(1 + a) \leq -n$, a contradiction. Thus, $v(a) \geq 0$ and $a \in V$. Hence, $m \subseteq V$.

Proof (continued)

So $b^n = 1 + a$.

Let v be the valuation associated to V. Then $nv(b) = v(1 + a)$. If $v(a) < 0$ then $v(1 + a) < 0$. Hence, $v(b) \leq -1$ and $v(1 + a) \leq -n$, a contradiction. Thus, $v(a) \geq 0$ and $a \in V$. Hence, $m \subseteq V$.

Let $c \in R$. Choose $d \in m$, $d \neq 0$ (possible as R is not a field). If $v(c) < 0$ then $v(c^\ell d) < 0$ for ℓ sufficiently large. But this contradicts $c^\ell d \in m \subseteq V$. Hence, $v(c) \geq 0$ and $R \subseteq V$. □

Localization and completion

Proposition

Let R be a Noetherian ring and p a prime of R. Suppose R_p is complete. Then either p is minimal or $R \cong R_p \times T$ for some Noetherian ring T.

Localization and completion

Proposition

Let R be a Noetherian ring and p a prime of R. Suppose R_p is complete. Then either p is minimal or $R \cong R_p \times T$ for some Noetherian ring T.

Proof: Suppose p is not minimal. It suffices to prove the natural map $\rho : R \to R_p$ is surjective. By replacing R with $R/\ker \rho$, we may assume $R \to R_p$ is injective.

Localization and completion

Proposition

Let R be a Noetherian ring and p a prime of R. Suppose R_p is complete. Then either p is minimal or $R \cong R_p \times T$ for some Noetherian ring T.

Proof: Suppose p is not minimal. It suffices to prove the natural map $\rho : R \to R_p$ is surjective. By replacing R with $R/\ker \rho$, we may assume $R \to R_p$ is injective.

Let $a \in R$, $a \notin p$. It suffices to show a is a unit, which can be checked in R/q for every minimal prime q. Hence, we may assume R is a domain, not a field (as p is not minimal).

Proof (continued)

If a is not a unit in R, then $a \in n$ for some maximal ideal n. There exists a DVR V with the same field of fractions as R such that $m_V \cap R = n$. Since R_p is complete, we must have $R_p \subseteq V$ by Kunz's Lemma.

Proof (continued)

If a is not a unit in R, then $a \in n$ for some maximal ideal n. There exists a DVR V with the same field of fractions as R such that $m_V \cap R = n$. Since R_p is complete, we must have $R_p \subseteq V$ by Kunz's Lemma.

But as a is a unit in R_p, a is also a unit in V, contradicting that $a \in n \subseteq m_V$. □

Proof of Theorem 2

Theorem 2

Let (R, m) be a local ring and M a reflexive R_p-module for some prime p. Suppose p is not minimal over $\operatorname{Ann}_R M$. Then M is reflexive as an R-module.

Proof of Theorem 2

Theorem 2

Let (R, m) be a local ring and M a reflexive R_p-module for some prime p. Suppose p is not minimal over $\operatorname{Ann}_R M$. Then M is reflexive as an R-module.

Proof: M is reflexive as an R-module if and only if M is reflexive as an $R/\operatorname{Ann}_R M$-module. Hence, we may assume $\operatorname{Ann}_R M = 0$.

Proof of Theorem 2

Theorem 2

Let (R, m) be a local ring and M a reflexive R_p-module for some prime p. Suppose p is not minimal over $\operatorname{Ann}_R M$. Then M is reflexive as an R-module.

Proof: M is reflexive as an R-module if and only if M is reflexive as an $R/\operatorname{Ann}_R M$-module. Hence, we may assume $\operatorname{Ann}_R M = 0$.

Hence, $\operatorname{Ann}_{R_p} M = 0$. In particular, the map $R \to R_p$ is injective.

Proof of Theorem 2

Theorem 2

Let (R, m) be a local ring and M a reflexive R_p-module for some prime p. Suppose p is not minimal over $\mathrm{Ann}_R M$. Then M is reflexive as an R-module.

Proof: M is reflexive as an R-module if and only if M is reflexive as an $R/\mathrm{Ann}_R M$-module. Hence, we may assume $\mathrm{Ann}_R M = 0$.

Hence, $\mathrm{Ann}_{R_p} M = 0$. In particular, the map $R \to R_p$ is injective.

By the BEGR Theorem, R_p is complete.

Proof of Theorem 2

Theorem 2

Let (R, m) be a local ring and M a reflexive R_p-module for some prime p. Suppose p is not minimal over $\operatorname{Ann}_R M$. Then M is reflexive as an R-module.

Proof: M is reflexive as an R-module if and only if M is reflexive as an $R/\operatorname{Ann}_R M$-module. Hence, we may assume $\operatorname{Ann}_R M = 0$.

Hence, $\operatorname{Ann}_{R_p} M = 0$. In particular, the map $R \to R_p$ is injective.

By the BEGR Theorem, R_p is complete.

By the Proposition, $R = R_p$, and M is reflexive as an R-module. □

The End

Thank you!