Logistic growth

October 30, 2013

Modeling the US population

Census data from 1790-2000

Year	Population	Year	Population	Year	Population
1790	3.9	1870	38.6	1950	150.7
1800	5.3	1880	50.2	1960	179.3
1810	7.2	1890	62.9	1970	203.3
1820	9.6	1900	76.0	1980	226.5
1830	12.9	1910	92.0	1990	248.7
1840	17.1	1920	105.7	2000	281.4
1850	23.1	1930	122.8		
1860	31.4	1940	131.7		

Modeling the US population

What is the population of the US on August 7, 2012?

Modeling the US population

- $314,160,921$

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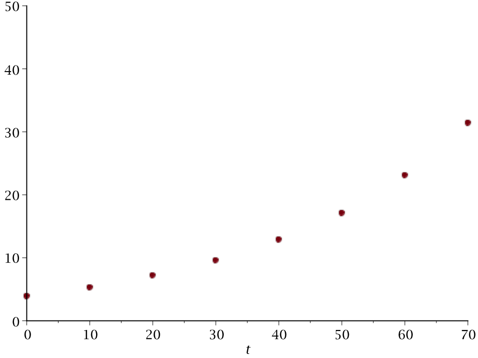
- $\approx \pi \times 100,000,000$

Modeling the US population

- $314,160,921$
- $\approx \pi \times 100,000,000$
- Pi-hundred-million population

Look like an exponential growth $P(t) = P_0 a^t$, t is in years since 1790.



The Years 1790-1860

What is P_0? What is a?

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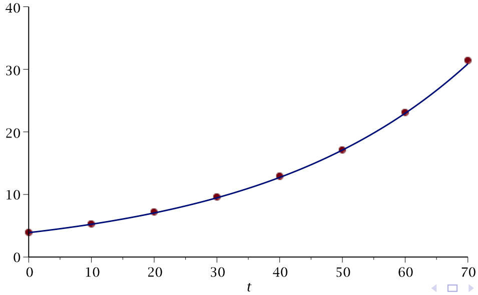
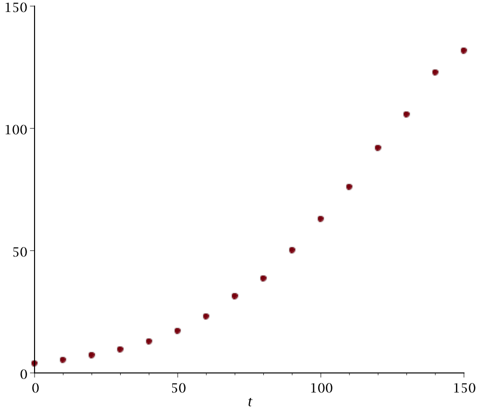
The Years 1790-1860

$$P(t) = 3.9(1.03)^t.$$



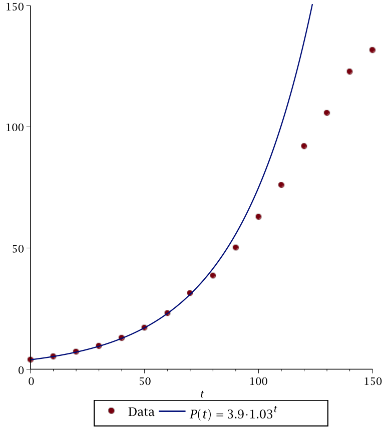
The Years 1790-1940

Can we still use the function $P(t) = 3.9(1.03)^t$ to approximate the population function?



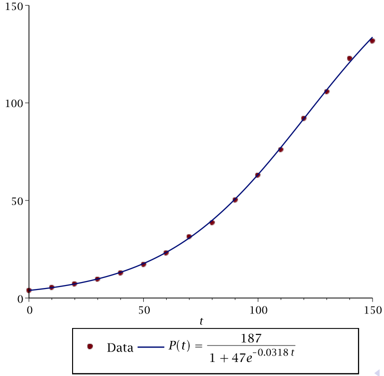
The Years 1790-1940

NO!



This kind of growth is modeled with a **logistic function**. If t is in years since 1790, then the function is

$$P(t) = \frac{187}{1 + 47e^{-0.0318t}}$$



Logistic function

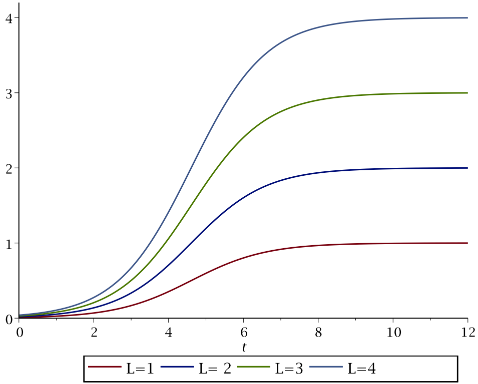
Definition

For positive constants L, C, and k, a logistic function has the form

$$P(t) = \frac{L}{1 + Ce^{-kt}}.$$

Example: $P = \frac{L}{1+100e^{-kt}}$.

Let $k = 1$.



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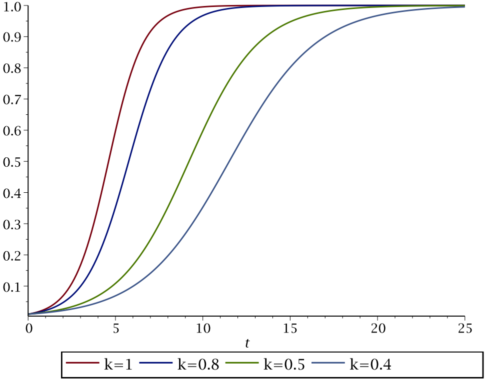

The Carrying Capacity and the Point of Diminishing Returns

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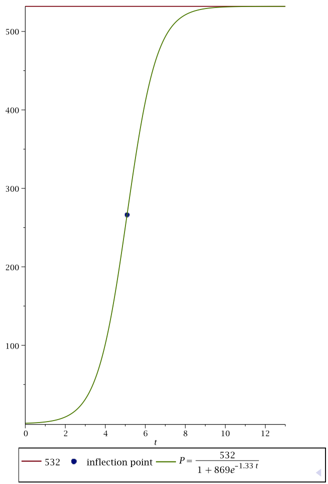
- The logistic function is approximately exponential function for small value of t, with grow rate k.

The Carrying Capacity and the Point of Diminishing Returns

Find the carrying capacity and the point of diminishing returns.



The Years 1790-2000: Another look at the US Population

- If t is in years since 1790 and P is in millions, we used the function
$$P(t) = \frac{187}{1 + 47e^{-0.0318t}}$$
to approximate the US population between 1790-1940.

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- According to this function, what is the maximum US population?
- Is this accurate?

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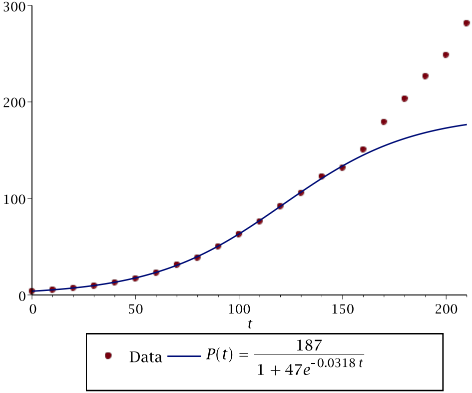

Sale Predictions

Total sale of a new product often follows a logistic model.

- Find the point where the concavity changes in the function. Use to estimate the maximum potential sale.

t (months)	0	1	2	3	4	5	6	7
P (total sales in 100s)	0.5	2	8	33	95	258	403	496

Sale Predictions

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- Find the point where the concavity changes in the function. Use to estimate the maximum potential sale.
- Using logistic regression, fit a logistic function to this data.

$$P = \frac{532}{1 + 869e^{-1.33t}}$$

What maximum potential sales does this function predict?

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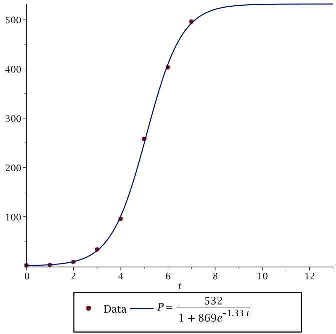
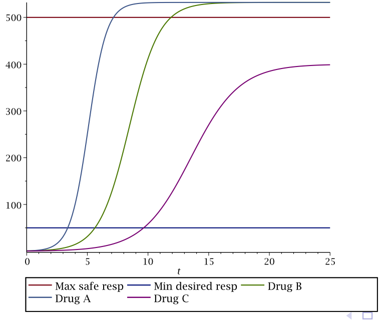
What maximum potential sales does this function predict?



Dose-Response Curves

- A dose-response curve plots the intensity of physiological response to a drug as a function of the dose administered.



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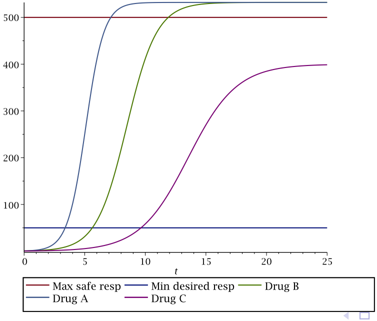

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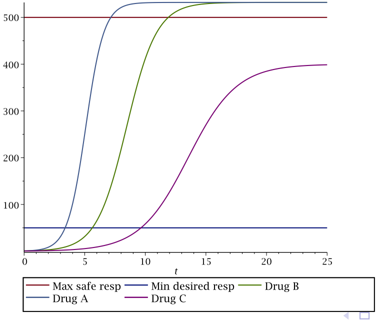

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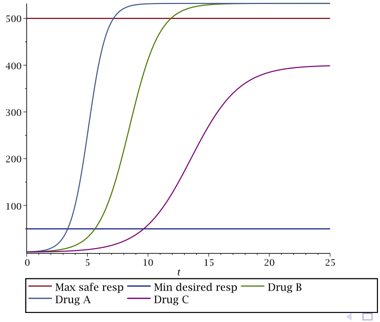
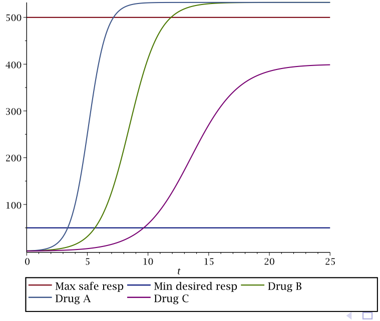
- Dose-response curves often follow a logistic model.



Dose-Response Curves

- Dose-response curve shows the amount of drug needed to produce the desired effect



Dose-Response Curves

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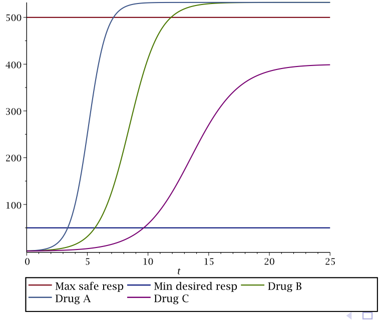
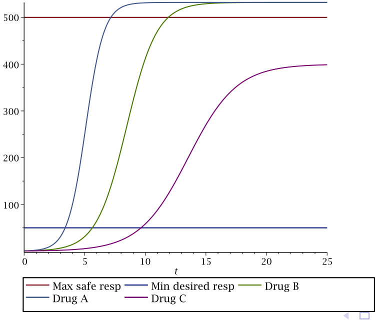
- Dose-response curve shows the amount of drug needed to produce the maximum effect attainable



Dose-Response Curves

- Dose-response curve shows the amount of drug needed to produce the desired effect
- Dose-response curve shows the amount of drug needed to produce the maximum effect attainable
- Drugs need to be administered in a dose which is large enough to be effective but not so large as to be dangerous.



Example

Discuss the advantages and disadvantages of three drugs in Figure.