The Definite Integral

November 4, 2013

Improving the Approximation

- In section 5.1, we saw how to approximate the total change given the rate of change. We now see how to make the approximation more accurate.

- In section 5.1, we saw how to approximate the total change given the rate of change. We now see how to make the approximation more accurate.
- We have seen that to have better estimation we need smaller interval of t (i.e. larger number of t-intervals, n). We will use notation Δt for the size of the t-interval used.

Example 1

If t is in hours since the start of a 20-hour period, a bacteria population increases at a rate given by

$$f(t) = 3 + 0.1t^2 \text{ millions of bacteria per hour.}$$

Make an underestimate of the total change in the number of bacteria over this period using

- (a) $\Delta t = 4$ hours

Example 1

If t is in hours since the start of a 20-hour period, a bacteria population increases at a rate given by

$$f(t) = 3 + 0.1t^2 \text{ millions of bacteria per hour.}$$

Make an underestimate of the total change in the number of bacteria over this period using

- (a) $\Delta t = 4$ hours
- (b) $\Delta t = 2$ hours

Example 1

If t is in hours since the start of a 20-hour period, a bacteria population increases at a rate given by

$$f(t) = 3 + 0.1t^2 \text{ millions of bacteria per hour.}$$

Make an underestimate of the total change in the number of bacteria over this period using

- (a) $\Delta t = 4$ hours
- (b) $\Delta t = 2$ hours
- (c) $\Delta t = 1$ hour

Example 1

If t is in hours since the start of a 20-hour period, a bacteria population increases at a rate given by

$$f(t) = 3 + 0.1t^2 \text{millions of bacteria per hour.}$$

Make an underestimate of the total change in the number of bacteria over this period using $\Delta t = 4$ hours.

- Make the table of the values of $f(t)$ at the time 0, Δt, $2 \cdot \Delta t$, $3 \cdot \Delta t, \ldots$

Example 1

If t is in hours since the start of a 20-hour period, a bacteria population increases at a rate given by

$$f(t) = 3 + 0.1t^2 \text{millions of bacteria per hour.}$$

Make an underestimate of the total change in the number of bacteria over this period using $\Delta t = 4$ hours.

- Make the table of the values of $f(t)$ at the time 0, Δt, $2 \cdot \Delta t$, $3 \cdot \Delta t, \ldots$
- Base on the table to find the underestimate.

Example 1

If t is in hours since the start of a 20-hour period, a bacteria population increases at a rate given by

$$f(t) = 3 + 0.1t^2 \text{millions of bacteria per hour.}$$

Make an underestimate of the total change in the number of bacteria over this period using $\Delta t = 4$ hours.

t (hours)	0	4	8	12	16	20
f(t)	3.0	4.6	9.4	17.4	28.6	43.0

Example 1

Total change $\approx 3.0 \cdot 4 + 4.6 \cdot 4 + 9.4 \cdot 4 + 17.4 \cdot 4 + 28.6 \cdot 4$ millions bacteria.

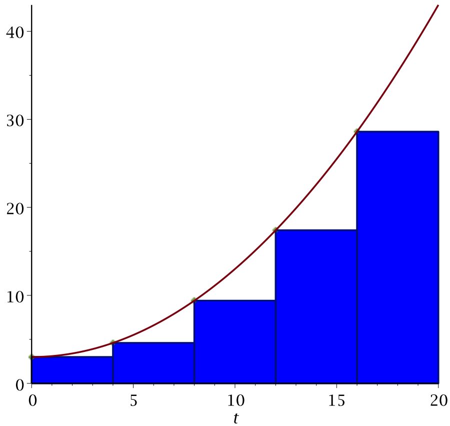
t (hours)	0	4	8	12	16	20
f(t)	3.0	4.6	9.4	17.4	28.6	43.0

Example 1

Total change $\approx 3.0 \cdot 4 + 4.6 \cdot 4 + 9.4 \cdot 4 + 17.4 \cdot 4 + 28.6 \cdot 4 = 252$ millions bacteria.

t (hours)	0	4	8	12	16	20
f(t)	3.0	4.6	9.4	17.4	28.6	43.0

Example 1



Example 1

If t is in hours since the start of a 20-hour period, a bacteria population increases at a rate given by

$$f(t) = 3 + 0.1t^2 \text{millions of bacteria per hour.}$$

Make an underestimate of the total change in the number of bacteria over this period using $\Delta t = 2$ hours.

t (hours)	0	2	4	6	8	10	12	14	16	18	20
f(t)	3.0	3.4	4.6	6.6	9.4	13.0	17.4	22.6	28.6	35.4	43.0

Example 1

Total change
$\approx 3.0 \cdot 2 + 3.4 \cdot 2 + 4.6 \cdot 2 + 6.6 \cdot 2 + 9.4 \cdot 2 + 13.0 \cdot 2 + 17.4 \cdot 2 +$
$+22.6 \cdot 2 + 28.6 \cdot 2 + 35.4 \cdot 2 + 43.0 \cdot 2$ millions bacteria.

t (hours)	0	2	4	6	8	10	12	14	16	18	20
f(t)	3.0	3.4	4.6	6.6	9.4	13.0	17.4	22.6	28.6	35.4	43.0

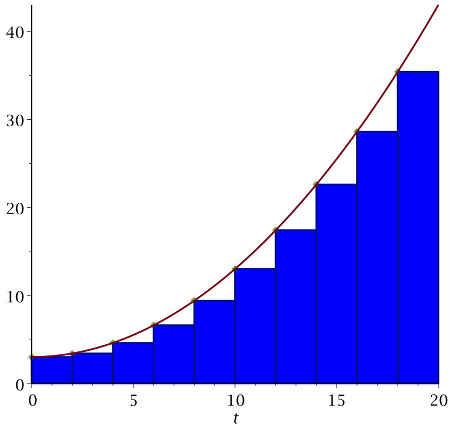
Example 1

Total change

$\approx 3.0 \cdot 2 + 3.4 \cdot 2 + 4.6 \cdot 2 + 6.6 \cdot 2 + 9.4 \cdot 2 + 13.0 \cdot 2 + 17.4 \cdot 2+$
$+22.6 \cdot 2 + 28.6 \cdot 2 + 35.4 \cdot 2 = 288$ millions bacteria.

t (hours)	0	2	4	6	8	10	12	14	16	18	20
f(t)	3.0	3.4	4.6	6.6	9.4	13.0	17.4	22.6	28.6	35.4	43.0

Example 1



Example 1

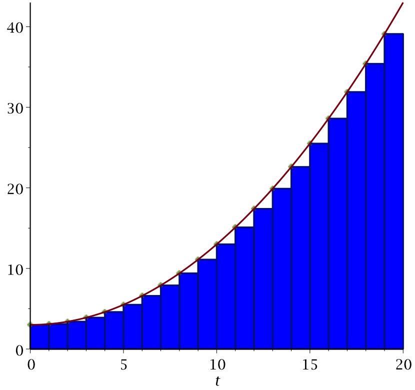
If t is in hours since the start of a 20-hour period, a bacteria population increases at a rate given by

$$f(t) = 3 + 0.1t^2 \text{millions of bacteria per hour.}$$

Make an underestimate of the total change in the number of bacteria over this period using $\Delta t = 1$ hours. We get total change is approximately 307 millions bacteria.

Example 1

As n larger, the estimate improves and the area of the shaded rectangles approaches the area under the curve.



Left- and Right-Hand Sums

- $f(t)$ is a continuous function for $a \leq t \leq b$.
- Divide the interval from a to b into n equal **subdivisions** (i.e. subintervals), each of width $\Delta t = \frac{b-a}{n}$.
- $a = t_0, t_1, \ldots, t_n = b$ are the endpoints of the subdivisions.

Left- and Right-Hand Sums

- $f(t)$ is a continuous function for $a \leq t \leq b$.
- Divide the interval from a to b into n equal **subdivisions** (i.e. subintervals), each of width $\Delta t = \frac{b-a}{n}$.
- $a = t_0, t_1, \ldots, t_n = b$ are the endpoints of the subdivisions.
- For a **left-hand sum**, we use the left endpoints of the subdivisions.

$$LHS = f(t_0)\Delta t + f(t_1)\Delta t + \ldots + f(t_{n-1})\Delta t$$

Left- and Right-Hand Sums

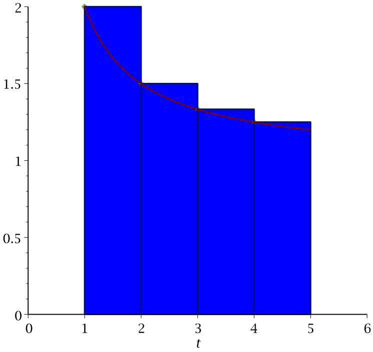
- $f(t)$ is a continuous function for $a \le t \le b$.
- Divide the interval from a to b into n equal **subdivisions** (i.e. subintervals), each of width $\Delta t = \frac{b-a}{n}$.
- $a = t_0, t_1, \ldots, t_n = b$ are the endpoints of the subdivisions.
- For a **left-hand sum**, we use the left endpoints of the subdivisions.

$$LHS = f(t_0)\Delta t + f(t_1)\Delta t + \ldots + f(t_{n-1})\Delta t$$

- For a **right-hand sum**, we use the right endpoints of the subdivisions.

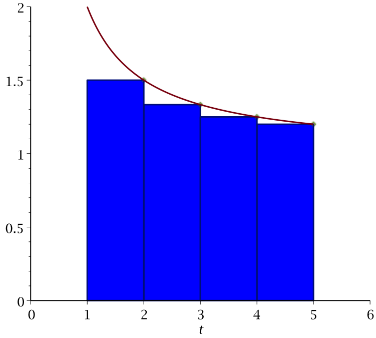
$$RHS = f(t_1)\Delta t + f(t_2)\Delta t + \ldots + f(t_n)\Delta t$$

Left- and Right-Hand Sums



The left-hand sum of f(t)=1+1/t from a=1 to b=5 and n=4

Left- and Right-Hand Sums



The right-hand sum of f(t)=1+1/t from a=1 to b=5 and n=
4

- $$LHS = \sum_{i=0}^{n-1} f(t_i)\Delta t = f(t_0)\Delta t + f(t_1)\Delta t + \ldots + f(t_{n-1})\Delta t$$

- $$LHS = \sum_{i=0}^{n-1} f(t_i)\Delta t = f(t_0)\Delta t + f(t_1)\Delta t + \ldots + f(t_{n-1})\Delta t$$
- $\sum$ tells us to add terms of the form $f(t_i)\Delta t$ for integers i running from 0 to $n-1$.

- $$LHS = \sum_{i=0}^{n-1} f(t_i)\Delta t = f(t_0)\Delta t + f(t_1)\Delta t + \ldots + f(t_{n-1})\Delta t$$
- $\sum$ tells us to add terms of the form $f(t_i)\Delta t$ for integers i running from 0 to $n-1$.
- $$RHS = f(t_1)\Delta t + f(t_2)\Delta t + \ldots + f(t_n)\Delta t$$

- $$LHS = \sum_{i=0}^{n-1} f(t_i)\Delta t = f(t_0)\Delta t + f(t_1)\Delta t + \ldots + f(t_{n-1})\Delta t$$
- $\sum$ tells us to add terms of the form $f(t_i)\Delta t$ for integer i runs from $i = 0$ to $i = n - 1$.
- $$RHS = f(t_1)\Delta t + f(t_2)\Delta t + \ldots + f(t_n)\Delta t = \sum_{i=1}^{n} f(t_i)\Delta t$$

Taking the Limit to Obtain the Definite Integral

- If f is a rate of change of some quantity, the the left-hand sum and right-hand sum approximate the total change in the quantity.

Taking the Limit to Obtain the Definite Integral

- If f is a rate of change of some quantity, the the left-hand sum and right-hand sum approximate the total change in the quantity.
- For most functions f, the approximation is improved by increasing the value of n (the number of subdivisions).

Taking the Limit to Obtain the Definite Integral

- If f is a rate of change of some quantity, the the left-hand sum and right-hand sum approximate the total change in the quantity.
- For most functions f, the approximation is improved by increasing the value of n (the number of subdivisions).
- To find the total change exactly, we take larger and larger values of n, and look at the values approached by the left-and right-hand sums.

Taking the Limit to Obtain the Definite Integral

- If f is a rate of change of some quantity, the the left-hand sum and right-hand sum approximate the total change in the quantity.
- For most functions f, the approximation is improved by increasing the value of n (the number of subdivisions).
- To find the total change exactly, we take larger and larger values of n, and look at the values approached by the left-and right-hand sums.
- This is called taking **limit** of these sums as n goes to infinity, and is written by

$$\lim_{n\to\infty} .$$

Taking the Limit to Obtain the Definite Integral

- If f is a rate of change of some quantity, the the left-hand sum and right-hand sum approximate the total change in the quantity.
- For most functions f, the approximation is improved by increasing the value of n (the number of subdivisions).
- To find the total change exactly, we take larger and larger values of n, and look at the values approached by the left-and right-hand sums.
- This is called taking **limit** of these sums as n goes to infinity, and is written by

$$\lim_{n\to\infty} .$$

- If f is continuous for $a \leq t \leq b$, then the limits of the left- and right-hand sums exist and are equal.

Taking the Limit to Obtain the Definite Integral

- If f is a rate of change of some quantity, the the left-hand sum and right-hand sum approximate the total change in the quantity.
- For most functions f, the approximation is improved by increasing the value of n (the number of subdivisions).
- To find the total change exactly, we take larger and larger values of n, and look at the values approached by the left-and right-hand sums.
- This is called taking **limit** of these sums as n goes to infinity, and is written by
$$\lim_{n\to\infty}.$$
- If f is continuous for $a \leq t \leq b$, then the limits of the left- and right-hand sums exist and are equal.
- The **definite integral** is the common limit of these sum.

Taking the Limit to Obtain the Definite Integral

Definition

Suppose f is a continuous function for $a \leq t \leq b$. The definite integral of f from a to b, written

$$\int_a^b f(t)dt,$$

is the limit of the left- and right-hand sums with n subdivisions of $[a, b]$ as n get arbitrarily large.

Taking the Limit to Obtain the Definite Integral

Definition

If $t_0, t_1, \ldots, t_n$ are the endpoints of the subdivisions,

$$\int_a^b f(t)dt = \lim_{n\to\infty}(LHS) = \lim_{n\to\infty}\left(\sum_{i=0}^{n-1} f(t_i)\Delta t\right)$$

and

$$\int_a^b f(t)dt = \lim_{n\to\infty}(RHS) = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(t_i)\Delta t\right).$$

Each of these sums is called **Riemann sum**, f is called the **integrand**, and a and b are called the **limits of integration**.

Compute a Definite Integral

- In practice, we approximate a definite integral numerically using calculators or computers.

Compute a Definite Integral

- In practice, we approximate a definite integral numerically using calculators or computers.
- They compute sums for larger and larger values of n, and eventually give us a value for the integral.

Compute a Definite Integral

- In practice, we approximate a definite integral numerically using calculators or computers.
- They compute sums for larger and larger values of n, and eventually give us a value for the integral.
- Different calculators and computers may give sightly different estimations, owning to round-off error and the fact they may use different approximation methods.

Compute a Definite Integral

- In practice, we approximate a definite integral numerically using calculators or computers.
- They compute sums for larger and larger values of n, and eventually give us a value for the integral.
- Different calculators and computers may give sightly different estimations, owning to round-off error and the fact they may use different approximation methods.
- For example, using a calculator, we find $\int_1^3 t^2 dt = 8.667$ (exact value is $26/3$).

Compute a Definite Integral

- In practice, we approximate a definite integral numerically using calculators or computers.
- They compute sums for larger and larger values of n, and eventually give us a value for the integral.
- Different calculators and computers may give sightly different estimations, owning to round-off error and the fact they may use different approximation methods.
- For example, using a calculator, we find $\int_1^3 t^2 dt = 8.667$ (exact value is $26/3$).
- This integral represents the area between $t = 1$ and $t = 3$ under the curve $f(t) = t^2$.

Estimate a Definite Integral from a Table

We have estimated a definite integral (in terms of area under the curve) from a table many times. The general we have the method:

- Take left-hand sum from the table

Estimate a Definite Integral from a Table

We have estimated a definite integral (in terms of area under the curve) from a table many times. The general we have the method:

- Take left-hand sum from the table
- Take right-hand sum from the table

Estimate a Definite Integral from a Table

We have estimated a definite integral (in terms of area under the curve) from a table many times. The general we have the method:

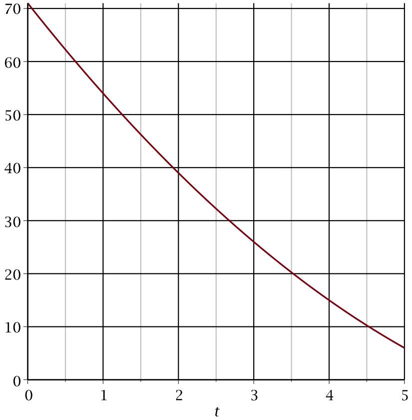
- Take left-hand sum from the table
- Take right-hand sum from the table
- The definite integral is approximately the average of the two sums.

Estimate a Definite Integral from a Table: Example

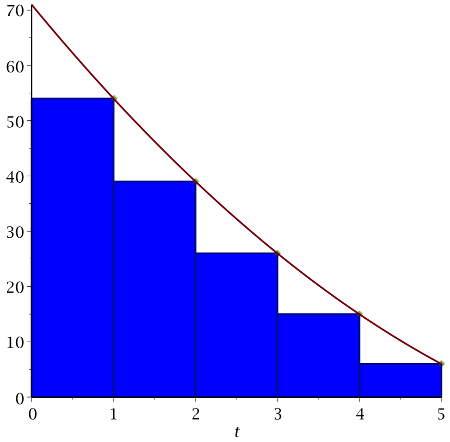
Estimate $\int_{20}^{30} f(t)dt$ from the table

t	20	22	24	26	28	30
f(t)	5	7	11	14	18	20

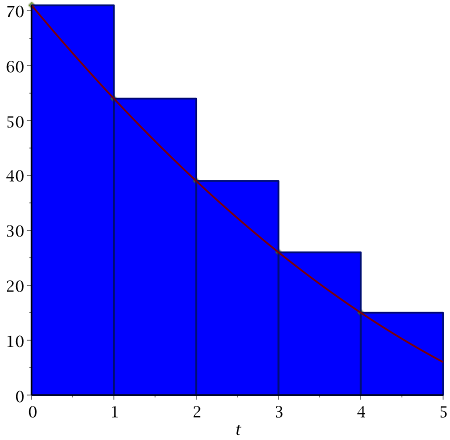
Estimate $\int_0^5 f(t)dt$ with the graph in the figure.



Estimate a Definite Integral from a Graph: Example

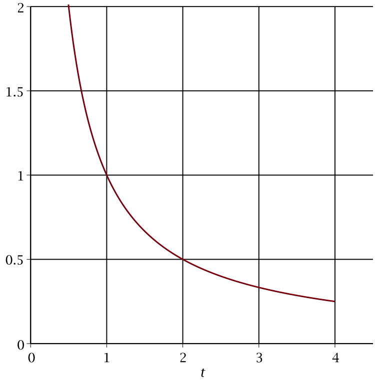


Estimate a Definite Integral from a Graph: Example



Roughly Estimation of a Definite Integral

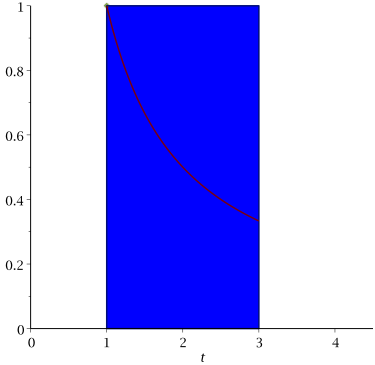
Three people calculated $\int_1^3 \frac{1}{t}dt$ on a calculator and got value 0.023, 11.984 and 1.526. Explain how you can be sure that none of these values is correct.



Roughly Estimation of a Definite Integral

Three people calculated $\int_1^3 \frac{1}{t} dt$ on a calculator and got value 0.023, 11.984 and 1.526.

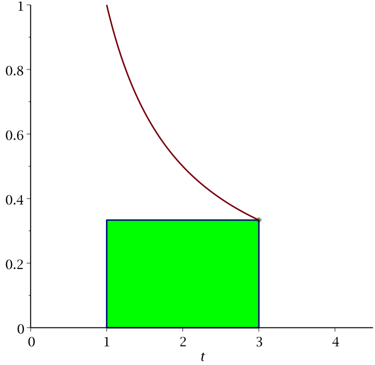
Overestimation is $2 \times 1 = 2$, thus the second answer is wrong.



Roughly Estimation of a Definite Integral

Three people calculated $\int_1^3 \frac{1}{t}dt$ on a calculator and got value 0.023, 11.984 and 1.526.

Underestimation is $2 \times 1/3 = 2/3$, so the first answer is wrong.



Roughly Estimation of a Definite Integral

Three people calculated $\int_1^3 \frac{1}{t} dt$ on a calculator and got value 0.023, 11.984 and 1.526.

Since the function is concave up, the area under the curve is less than $2 \times \frac{1}{3} + \frac{1}{2}\left(2 \times \frac{2}{3}\right) = 4/3$. Therefore, the last answer is also wrong.