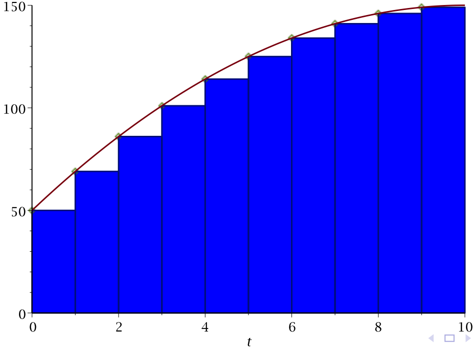
The Definite Integral as Area

November 6, 2013

The case when $f(x)$ is nonnegative

- Let $f(x)$ is nonnegative and continuous for $a \leq x \leq b$.



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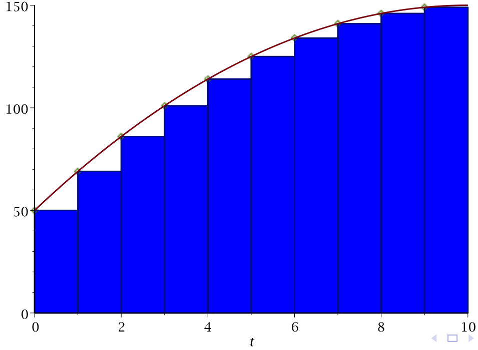

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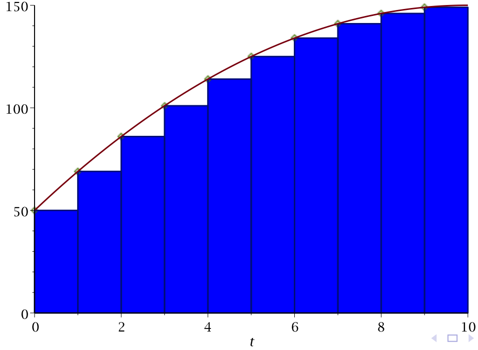

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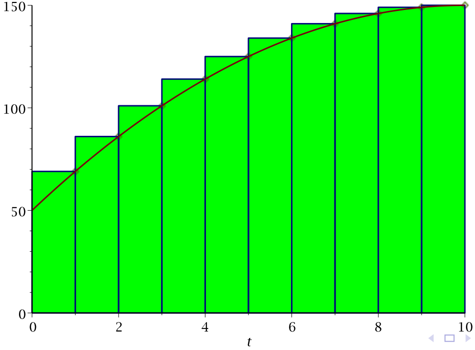

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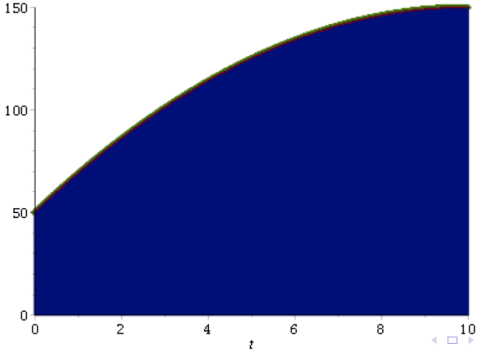

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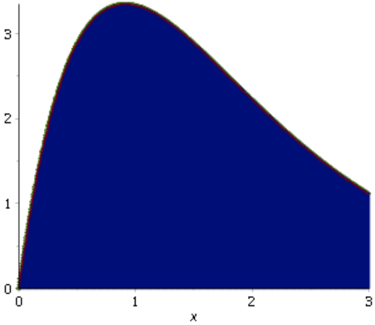
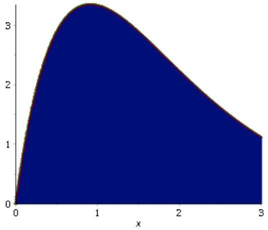


Area shaded $= \int_0^3 f(x)\,dx$, where $f(x) = 10\,x 3^{-x}$.

Example 1

Find the area under the graph of $y = 10x(3^{-x})$ between $x = 0$ and $x = 3$.

- Since the function is nonnegative for $0 \leq x \leq 3$, the area is equal to $\int_0^3 10x(3^{-x})dx$.



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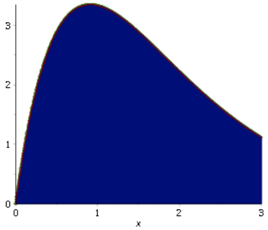
$$\int_0^3 10x(3^{-x})dx = 6.967$$



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Relationship Between Definite Integral and Area : When $f(x)$ is NOT nonnegative.

Consider the relationship between the definite integral $\int_{-1}^{1}(x^2 - 1)dx$ and the area between the parabola $y = x^2 - 1$ and the x-axis.

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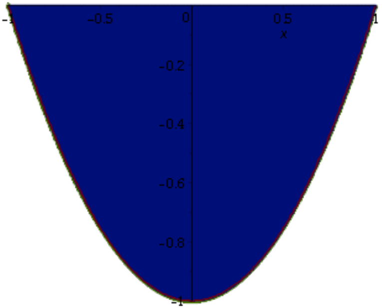

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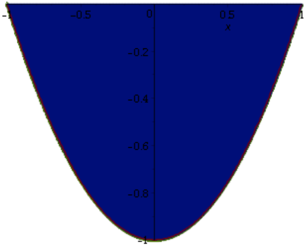
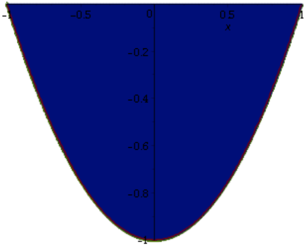
- The parabola lies below the x-axis between $x = -1$ and $x = 1$.



Integral $\int_{-1}^{1} f(x)\, dx$, where $f(x) = x^2 - 1$, is negative of shaded area.

Relationship Between Definite Integral and Area : When $f(x)$ is NOT nonnegative.

- The parabola lies below the x-axis between $x = -1$ and $x = 1$.
- $\int_{-1}^{1}(x^2 - 1)dx = -Area = -1.33$



Integral $\int_{-1}^{1} f(x)\, dx$, where $f(x) = x^2 - 1$, is negative of shaded area.

Conclusion

When $f(x)$ is positive for some x-values and negative for others, and $a < b$:
$\int_a^b f(x)dx$ is the sum of the areas above the x-axis, **counted positively**, and the areas below the x-axis, **counted negatively**.

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Interpret the definite integral $\int_0^4 (x^3 - 7x^2 + 11x)dx$ in terms of areas.

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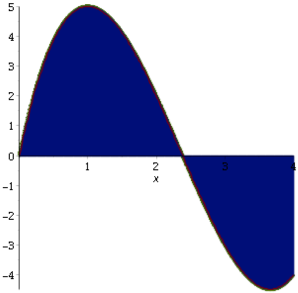
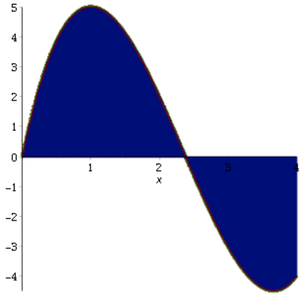


Example 3

- the graph of $f(x)$ is crossing below the x-axis about $x = 2.4$.



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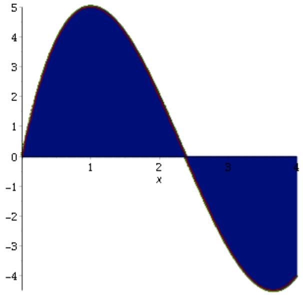

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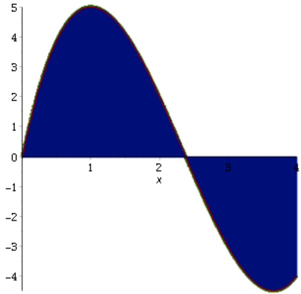

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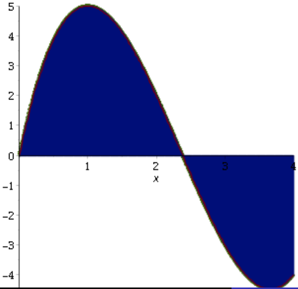

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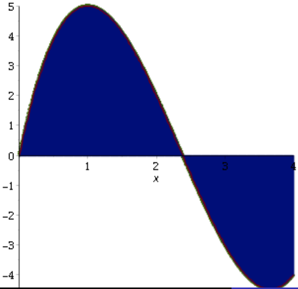

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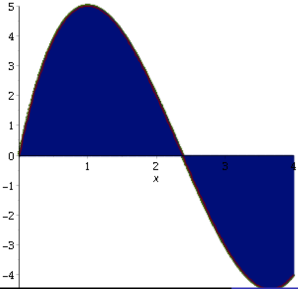

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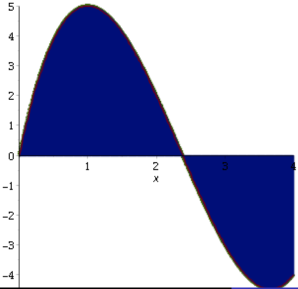
- Use calculator $\int_0^4 (x^3 - 7x^2 + 11x)dx = 2.67$
- $\int_0^4 (x^3 - 7x^2 + 11x)dx = A_1 - A_2$.
- Break the integral into two parts
  $A_1 \approx \int_0^{2.4} (x^3 - 7x^2 + 11x)dx \approx 7.72$ and
  $-A_2 \approx \int_{2.4}^4 (x^3 - 7x^2 + 11x)dx \approx -5.05$.



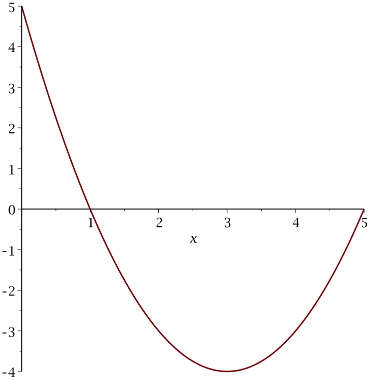
Property

Assume $f(x)$ is continuous from a to b, and $a < c < b$. Then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Example 4

For each functions graphed in the figure, decide whether $\int_0^5 f(x)dx$ is positive, negative or approximately zero.



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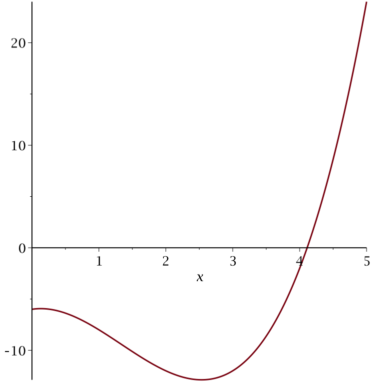

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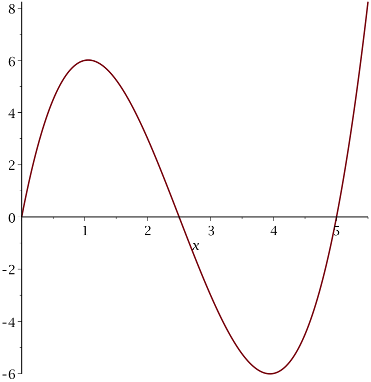
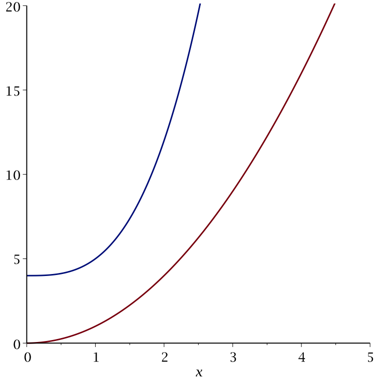


Area between two curves



Area between two curves

If $g(x) \leq f(x)$ for $a \leq x \leq b$. Then
Area between graphs of $f(x)$ and $g(x)$ for $a \leq x \leq b$ is equal to
$\int_a^b (f(x) - g(x))dx$.

Formula

Assume $f(x)$ and $g(x)$ are two continuous functions for $a \leq x \leq b$, and A and B are two constants. Then

$$\int_a^b (Af(x) + Bg(x))dx = A\int_a^b f(x)dx + B\int_a^b g(x)dx$$

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Graph $f(x) = 4x - x^2$ and $g(x) = \frac{1}{2}x^{3/2}$ for $x \geq 0$. Use a definite integral to estimate the area enclosed by the graphs of these two functions.

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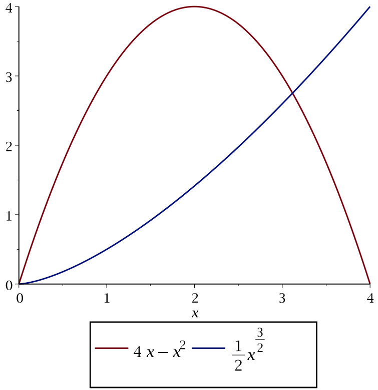

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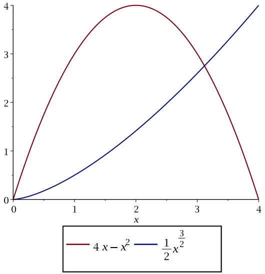

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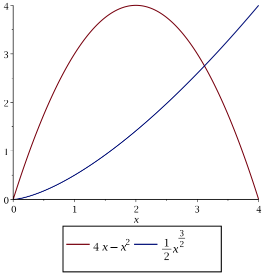
- Between $x = 0$ and $x = 3.1$ the graph of $f(x) = 4x - x^2$ is above the graph of $g(x) = \frac{1}{2}x^{3/2}$.



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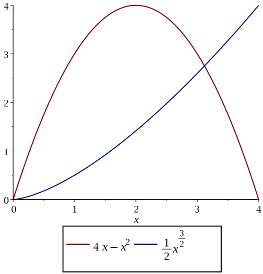
- Between $x = 0$ and $x = 3.1$ the graph of $f(x) = 4x - x^2$ is above the graph of $g(x) = \frac{1}{2}x^{3/2}$.
- The Area is (approximately) $\int_0^{3.1} (4x - x^2 - \frac{1}{2}x^{3/2})dx$.



Area between two curves

For $a \leq x \leq b$
$\int_a^b (f(x) - g(x))dx$ is the sum of the areas when $f(x) > g(x)$, counted positively, and the areas when $f(x) < g(x)$, counted negatively.

Area between two curves

$$\int_0^{10} (\sin(x) - \sin(-x))dx$$