Integration by Parts

December 4, 2013

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- NEED A MORE POWERFUL METHOD!

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- Integrate both sides: $\int uv'dx = \int (uv)'dx - \int u'vdx$
- Since $\int (uv)'dx = uv$, we get

$$\int uv'dx = uv - \int u'vdx$$

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- Thus $u' = 1$, $v = e^{2w}/2$.
- $$\begin{aligned} \int uv' dw = \int e^{2w} w dw = uv - \int u' v dx \\ = w(e^{2w}/2) - \int (1)(e^{2w}/2) dw \\ = w(e^{2w}/2) - \frac{1}{2} \int e^{2w} dw \\ = we^{2w}/2 - \frac{1}{2}(e^{2w}/2) + C \end{aligned}$$

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- Let $u = \ln x$ and $v' = x$.
- $u' = 1/x$, and $v = x^2/2$.
- $$\begin{aligned}\int uv'dx = \int x \ln x dx &= uv - \int u'v dx \\ &= (\ln x)(x^2/2) - \int (1/x)(x^2/2) dx \\ &= (x^2 \ln x)/2 - \int x/2 dx = (x^2 \ln x)/2 - x^2/4 + C\end{aligned}$$

How to chose u and v'

Consider again example

$$\int e^{2w} w dw = ?$$

What happen if we chose $u = e^{2w}$ and $v' = w$?

- We get $u' = 2e^{2w}$ and $v = w^2/2$.

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$$\int e^{2w} w dw = uv - \int u' v dw = e^{2w}(w^2/2) - \int 2e^{2w}(w^2/2) dw$$

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- Finding $\int 2e^{2w}(w^2/2)dw$ is even harder than finding the original integral.
- The choice: $u = e^{2w}$ and $v' = w$ doesn't work!

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- We need to make sure uv' is equal to the integrand.
- Whatever we let v' be, we need to find v (i.e. find the antiderivative of the function v').
- It helps if $u'v$ is simpler (or least no more compilated) than uv'.

Example

- $$\int \ln x dx = ?$$

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- Let $u = \ln x$ and $v' = 1$.

Example

$$\int x^8 \ln x dx = ?$$

Example

$$\int \frac{ln x}{x^2} dx = ?$$

Example

$$\int (x+2)\sqrt{2+3x}\,dx = ?$$

Example

$$\int t^2 e^{5t} dt = ?$$

Example

$$\int ze^{-z}dz = ?$$

Example

$$\int \frac{x}{\sqrt{5x+2}} dx = ?$$

Example

$$\int xe^{x}e^{e^{x}}\,dx = ?$$