Average Values, Present and Future Values

November 18, 2013

The definite integral as an Average

- Assume the temperature in 24-hour period is measured by a function $f(t)$, where t is in hours since the beginning of the period.

The definite integral as an Average

- Assume the temperature in 24-hour period is measured by a function $f(t)$, where t is in hours since the beginning of the period.
- We consider the temperature at n equally spaced times $t_1, t_2, \ldots, t_n$. The average temperature is $\frac{f(t_1)+f(t_2)+\ldots+f(t_n)}{n}$.

The definite integral as an Average

- Assume the temperature in 24-hour period is measured by a function $f(t)$, where t is in hours since the beginning of the period.
- We consider the temperature at n equally spaced times $t_1, t_2, \ldots, t_n$. The average temperature is $\frac{f(t_1)+f(t_2)+\ldots+f(t_n)}{n}$.
- Assume the length of the subinterval is Δt, then $n = 24/\Delta t$ and the average temperature is $\frac{f(t_1)+f(t_2)+\ldots+f(t_n)}{24/\Delta t}$.

The definite integral as an Average

- Assume the temperature in 24-hour period is measured by a function $f(t)$, where t is in hours since the beginning of the period.
- We consider the temperature at n equally spaced times $t_1, t_2, \ldots, t_n$. The average temperature is $\frac{f(t_1)+f(t_2)+\ldots+f(t_n)}{n}$.
- Assume the length of the subinterval is Δt, then $n = 24/\Delta t$ and the average temperature is $\frac{f(t_1)+f(t_2)+\ldots+f(t_n)}{24/\Delta t}$.
- $$\textit{The average temperature} = \frac{\sum_{i=1}^{n} f(t_i)\Delta t}{24} = \frac{RHS}{24}$$

The definite integral as an Average

- Assume the temperature in 24-hour period is measured by a function $f(t)$, where t is in hours since the beginning of the period.
- We consider the temperature at n equally spaced times $t_1, t_2, \ldots, t_n$. The average temperature is $\frac{f(t_1)+f(t_2)+\ldots+f(t_n)}{n}$.
- Assume the length of the subinterval is Δt, then $n = 24/\Delta t$ and the average temperature is $\frac{f(t_1)+f(t_2)+\ldots+f(t_n)}{24/\Delta t}$.
- $$\textit{The average temperature} = \frac{\sum_{i=1}^{n} f(t_i)\Delta t}{24} = \frac{RHS}{24}$$
- Let $\Delta t \to 0$, we get $\textit{The average temperature} = \frac{\int_0^{24} f(t)dt}{24}$.

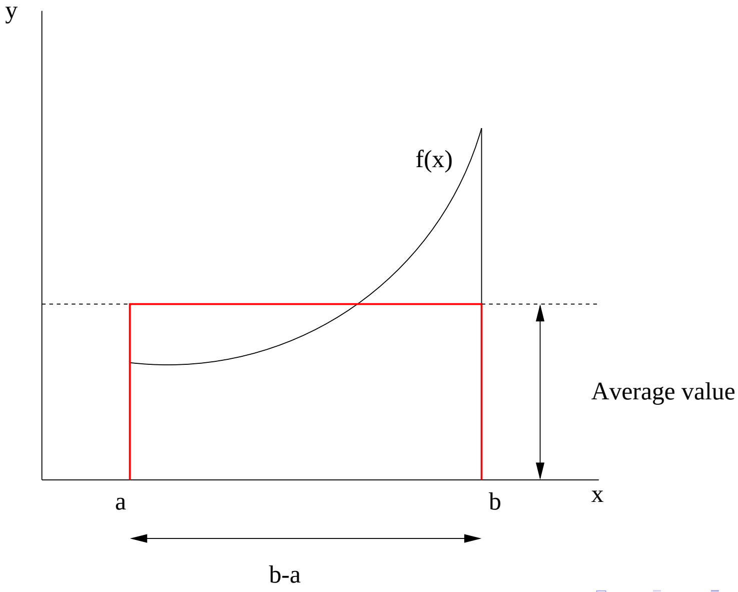
The definite integral as an Average

Definition

The Average value of a function $f(x)$ on the interval from a to b is defined to be

$$\frac{1}{b-a}\int_a^b f(t)dt$$

Visualize the average value



Example

Suppose that $C(t)$ represents the daily cost of heating your house, in dollars per day, where t is time in days and $t = 0$ corresponds to Jan 1, 2013. Then $\frac{1}{100} \int_0^{100} C(t) dt$ present the average cost per day to heat your house during the first 100 days of 2013.

Example

The population of McAllen, Texas can be modeled by the function

$$P(t) = 570(1.037)^t$$

where t is in years since 2000.

- Predict the average population of McAllen between the years 2020 and 2040.

Example

The population of McAllen, Texas can be modeled by the function

$$P(t) = 570(1.037)^t$$

where t is in years since 2000.

- Predict the average population of McAllen between the years 2020 and 2040.
- $\frac{1}{40-20}\int_{20}^{40} P(t)dt = \frac{1}{40-20}\int_{20}^{40} 570(1.037)^t dt$

Example

The population of McAllen, Texas can be modeled by the function

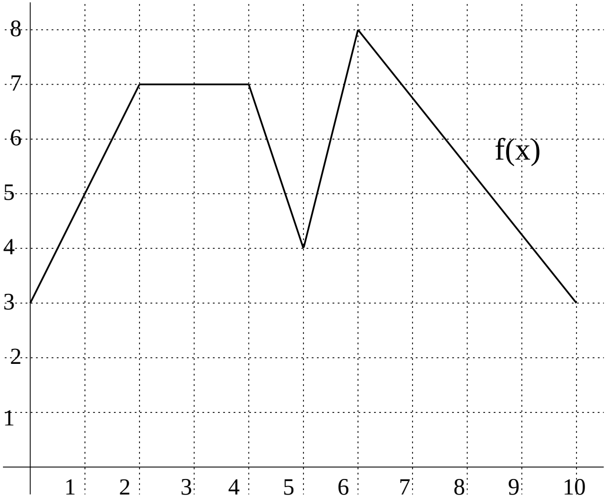
$$P(t) = 570(1.037)^t$$

where t is in years since 2000. Predict the average population of McAllen between the years 2020 and 2040.

- $\frac{1}{40-20}\int_{20}^{40} P(t)dt = \frac{1}{40-20}\int_{20}^{40} 570(1.037)^t dt$
- $= \frac{1}{20}(34,656)$

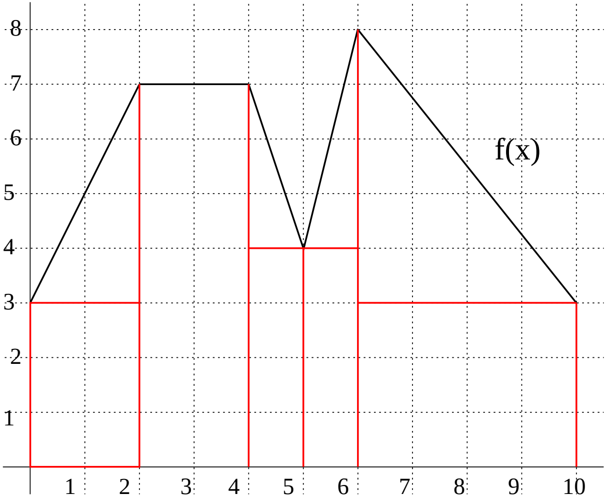
Example

Find the average of function $f(x)$ given in the figure on the interval $x = 0$ to $x = 10$.



Example

Find the average of function $f(x)$ given in the figure on the interval $x = 0$ to $x = 10$.



Income Stream

1. When we consider payments made to or by an individual, we usually think of **discrete** payments (i.e. payments made at specific moments in time).
2. However, we may think of payments made by a company as being **continuous**.
3. For example, the revenues earned by a huge corporation come essentially all the time, and therefore they can be represented by a continuous **income stream**.
4. Since the rate at which revenue is earned may vary from time to time, the income stream is described by

$$S(t) \text{ dollars/year.}$$

5. $S(t)$ is a rate at which payments are made, and that the rate depends on the time t, usually measured in years from the present.

Present and Future values of an income stream

Jus as we can find the present and future values of a single payment, so we can find the present and future values of an income stream.

- The **future value** presents the total amount of money that you would have if you deposited an income stream into a bank account as you receive it and let it earn interest until that future date.

Present and Future values of an income stream

Jus as we can find the present and future values of a single payment, so we can find the present and future values of an income stream.

- The **future value** presents the total amount of money that you would have if you deposited an income stream into a bank account as you receive it and let it earn interest until that future date.
- The **present value** presents the amount of money you would have to deposit today (in an interest-bearing bank account) in order to match what you would get from the income stream by that future date.

Present and Future values of an income stream

Jus as we can find the present and future values of a single payment, so we can find the present and future values of an income stream.

- The **future value** presents the total amount of money that you would have if you deposited an income stream into a bank account as you receive it and let it earn interest until that future date.
- The **present value** presents the amount of money you would have to deposit today (in an interest-bearing bank account) in order to match what you would get from the income stream by that future date.
- When we work with an income stream we always assume that the interest is compounded continuously.

Present and Future values of an income stream

Jus as we can find the present and future values of a single payment, so we can find the present and future values of an income stream.

- The **future value** presents the total amount of money that you would have if you deposited an income stream into a bank account as you receive it and let it earn interest until that future date.
- The **present value** presents the amount of money you would have to deposit today (in an interest-bearing bank account) in order to match what you would get from the income stream by that future date.
- When we work with an income stream we always assume that the interest is compounded continuously.
- $P = Be^{-rt}$.

Present and Future values of an income stream

Suppose that we want to calculate the present value of the income stream described by a rate of $S(t)$ dollars per year, and that we are interested in the period from now to M years in the future.

- We divide the stream into small deposits, and imagine each deposited at one instant.

Present and Future values of an income stream

Suppose that we want to calculate the present value of the income stream described by a rate of $S(t)$ dollars per year, and that we are interested in the period from now to M years in the future.

- We divide the stream into small deposits, and imagine each deposited at one instant.
- Dividing the inter val $0 \leq t \leq M$ into subintervals of length Δt.

Present and Future values of an income stream

Suppose that we want to calculate the present value of the income stream described by a rate of $S(t)$ dollars per year, and that we are interested in the period from now to M years in the future.

- We divide the stream into small deposits, and imagine each deposited at one instant.
- Dividing the inter val $0 \leq t \leq M$ into subintervals of length Δt.
- Since Δt is small, the rate $S(t)$ doesn't vary much within each subinterval. Thus between t and $t + \Delta t$:

$$\begin{aligned} \mathit{Amount\ paid} &\approx \mathit{Rate\ of\ deposit} \times \mathit{Time} \\ &\approx S(t)\Delta t\ \mathit{dollars} \end{aligned}$$

Present and Future values of an income stream

Suppose that we want to calculate the present value of the income stream described by a rate of $S(t)$ dollars per year, and that we are interested in the period from now to M years in the future.

- We divide the stream into small deposits, and imagine each deposited at one instant.
- Dividing the inter val $0 \leq t \leq M$ into subintervals of length Δt.
- Since Δt is small, the rate $S(t)$ doesn't vary much within each subinterval. Thus between t and $t + \Delta t$:

$$\begin{aligned} \textit{Amount paid} &\approx \textit{Rate of deposit} \times \textit{Time} \\ &\approx S(t)\Delta t \textit{ dollars} \end{aligned}$$

- The deposit $S(t)\Delta t$ is made t years in the future. Thus, the present value of money deposited in the interval t to $t + \Delta t$ is approximately $S(t)\Delta t e^{-rt}$.

Present and Future values of an income stream

- Since Δt is small, the rate $S(t)$ doesn't vary much within each subinterval. Thus between t and $t + \Delta t$:

$$\begin{aligned} \textit{Amount paid} &\approx \textit{Rate of deposit} \times \textit{Time} \\ &\approx S(t)\Delta t \textit{ dollars} \end{aligned}$$

- The deposit $S(t)\Delta t$ is made t years in the future. Thus, the present value of money deposited in the interval t to $t + \Delta t$ is approximately $S(t)\Delta t e^{-rt}$.

Present and Future values of an income stream

- Since Δt is small, the rate $S(t)$ doesn't vary much within each subinterval. Thus between t and $t + \Delta t$:

$$\begin{aligned} Amount\ paid &\approx Rate\ of\ deposit \times Time \\ &\approx S(t)\Delta t\ dollars \end{aligned}$$

- The deposit $S(t)\Delta t$ is made t years in the future. Thus, the present value of money deposited in the interval t to $t + \Delta t$ is approximately $S(t)\Delta t e^{-rt}$.
- Summing over all subintervals gives:

$$Total\ present\ value \approx \sum S(t)e^{-rt}\Delta t\ dollars.$$

Present and Future values of an income stream

- Since Δt is small, the rate $S(t)$ doesn't vary much within each subinterval. Thus between t and $t + \Delta t$:

$$\begin{aligned} Amount\ paid\ &\approx Rate\ of\ deposit \times Time \\ &\approx S(t)\Delta t\ dollars \end{aligned}$$

- The deposit $S(t)\Delta t$ is made t years in the future. Thus, the present value of money deposited in the interval t to $t + \Delta t$ is approximately $S(t)\Delta t e^{-rt}$.
- Summing over all subintervals gives:

$$Total\ present\ value\ \approx \sum S(t)e^{-rt}\Delta t\ dollars.$$

- Let $\Delta t \to 0$,

$$Total\ present\ value\ = \int_0^M S(t)e^{-rt}dt.$$

Example

Find the present and future values of a constant income stream of \$1000 per year over a period of 20 years, assuming an interest rate of 6% compounded continuously.

Example

Find the present and future values of a constant income stream of \$1000 per year over a period of 20 years, assuming an interest rate of 6% compounded continuously.

- $Present\ \ value\ \ = \int_0^{20} 1000e^{-0.06t}dt$

Example

Find the present and future values of a constant income stream of \$1000 per year over a period of 20 years, assuming an interest rate of 6% compounded continuously.

- *Present value* $= \int_0^{20} 1000e^{-0.06t}\,dt = \$11.647$

Example

Find the present and future values of a constant income stream of \$1000 per year over a period of 20 years, assuming an interest rate of 6% compounded continuously.

- $Present\ \ value\ = \int_0^{20} 1000e^{-0.06t}dt = \$11.647$
- The future value is $\$11,647 \times e^{0.06(20)} = \$38,669$

Example

Find the present and future values of a constant income stream of \$1000 per year over a period of 20 years, assuming an interest rate of 6% compounded continuously.

- $\textit{Present value} = \int_0^{20} 1000e^{-0.06t}\,dt = \$11,647$

Example

Find the present and future values of a constant income stream of \$1000 per year over a period of 20 years, assuming an interest rate of 6% compounded continuously.

- $\textit{Present value} = \int_0^{20} 1000e^{-0.06t}dt = \$11,647$
- The future value is $\$11,647 \times e^{0.06(20)} = \$38,669$

Example

Find the present and future values of a constant income stream of \$1000 per year over a period of 20 years, assuming an interest rate of 6% compounded continuously.

- *Present value* $= \int_0^{20} 1000e^{-0.06t}\,dt = \$11,647$
- The future value is $\$11,647 \times e^{0.06(20)} = \$38,669$
- Total amount deposited is \$20,000.

Example

Find the present and future values of a constant income stream of \$1000 per year over a period of 20 years, assuming an interest rate of 6% compounded continuously.

- *Present value* $= \int_0^{20} 1000e^{-0.06t}dt = \$11,647$
- The future value is $\$11,647 \times e^{0.06(20)} = \$38,669$
- Total amount deposited is \$20,000.
- The money is almost doubled because of the interest.

Example

Suppose you want to have \$50,000 in 8 years in a bank account earning 3% interest compounded continuously.

Example: Part (a)

Suppose you want to have \$50,000 in 8 years in a bank account earning 3% interest compounded continuously. If you make one lump sum deposit now, how should you deposit?

- Assume the present value is P, then $B = 50,000$ is the future value.

Example: Part (a)

Suppose you want to have \$50,000 in 8 years in a bank account earning 3% interest compounded continuously. If you make one lump sum deposit now, how should you deposit?

- Assume the present value is P, then $B = 50,000$ is the future value.
- $B = 50,000 = Pe^{0.03(8)}$

Example: Part (a)

Suppose you want to have \$50,000 in 8 years in a bank account earning 3% interest compounded continuously. If you make one lump sum deposit now, how should you deposit?

- Assume the present value is P, then $B = 50,000$ is the future value.
- $B = 50,000 = Pe^{0.03(8)}$
- $P = \frac{50,000}{e^{0.24}}$

Example:Part (a)

Suppose you want to have \$50,000 in 8 years in a bank account earning 3% interest compounded continuously. If you make one lump sum deposit now, how should you deposit?

- Assume the present value is P, then $B = 50,000$ is the future value.
- $B = 50,000 = Pe^{0.03(8)}$
- $P = \frac{50,000}{e^{0.24}} = 39,331$

Suppose you want to have \$50,000 in 8 years in a bank account earning 3% interest compounded continuously. If the deposit money continuously throughout the 8-year period, at what (constant) rate should you deposit it?

- Assume the rate is a constant S.

Example:Part (b)

Suppose you want to have \$50,000 in 8 years in a bank account earning 3% interest compounded continuously. If the deposit money continuously throughout the 8-year period, at what (constant) rate should you deposit it?

- Assume the rate is a constant S.
- $Present\ value\ = \int_0^8 e^{-0.03t} dt$

Example:Part (b)

Suppose you want to have \$50,000 in 8 years in a bank account earning 3% interest compounded continuously. If the deposit money continuously throughout the 8-year period, at what (constant) rate should you deposit it?

- Assume the rate is a constant S.
- $Present\ \ value\ = \int_0^8 e^{-0.03t} dt$
- $Present\ \ value\ = S \int_0^8 e^{-0.03t} dt$

Example:Part (b)

Suppose you want to have \$50,000 in 8 years in a bank account earning 3% interest compounded continuously. If the deposit money continuously throughout the 8-year period, at what (constant) rate should you deposit it?

- Assume the rate is a constant S.
- $Present\ \ value\ = \int_0^8 e^{-0.03t} dt$
- $Present\ \ value\ = S \int_0^8 e^{-0.03t} dt = S(7.1124)$

Example:Part (b)

Suppose you want to have \$50,000 in 8 years in a bank account earning 3% interest compounded continuously. If the deposit money continuously throughout the 8-year period, at what (constant) rate should you deposit it?

- Assume the rate is a constant S.
- $Present\ value\ = \int_0^8 e^{-0.03t} dt$
- $Present\ value\ = S \int_0^8 e^{-0.03t} dt = S(7.1124)$
- Since present value of the continuous deposit must be the same as the present value of the lump sum deposit; that is $39,331$

$$39,331 = S(7.1124)$$

Example:Part (b)

Suppose you want to have \$50,000 in 8 years in a bank account earning 3% interest compounded continuously. If the deposit money continuously throughout the 8-year period, at what (constant) rate should you deposit it?

- Assume the rate is a constant S.
- $Present \ \ value \ \ = \int_0^8 e^{-0.03t} dt$
- $Present \ \ value \ \ = S \int_0^8 e^{-0.03t} dt = S(7.1124)$
- Since present value of the continuous deposit must be the same as the present value of the lump sum deposit; that is $39,331$

$$39,331 = S(7.1124)$$

- $S = \frac{39,331}{7.1124} = 5530.$