Chain rule

October 2, 2013

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- $f(t) = ln(3t+1) \rightarrow f(u) = \ln u$
- $y = 5(2x^2+4)^2 \rightarrow y = 5u^2$
- $P = (3x+1) + 4e^{9x+3} \rightarrow P = u + 4e^{3u}$

Derivative of a composite functions

- Suppose $y = f(z)$ and $z = g(t)$ for some outside function f and inside function g, where f and g are *differentiable*. A small change in t, Δt, generates a small change in z, Δz. In turn, Δz generates a small change in y, Δy. Provided Δt and Δz are not zero, we have

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- The derivative $\frac{dy}{dz}$ is the limit of the quotient $\frac{\Delta y}{\Delta z}$ as Δz gets smaller and smaller.
- $\frac{dy}{dt} = \frac{dy}{dz} \cdot \frac{dz}{dt}$.

Chain rule

Theorem

Let $y = f(z)$ and $z = g(t)$ be two differentiable functions, then the derivative of the composite function $y = f(g(t))$ is given by

$$\frac{dy}{dt} = \frac{dy}{dz} \cdot \frac{dz}{dt}.$$

In other word, the derivative of a composite fucntion is the derivative of the outside function times the derivative of the inside function:

$$\frac{d}{dt}(f(g(t))) = f'(g(t)) \cdot g'(t).$$

Example

- The amount of gas G, in gallons, consumed by a car depends on the distance traveled, s, in miles, and s depends on the time t, in hours. If 0.05 gallons of gas is consumed for each mile traveled, and the car is traveling at 30 miles/hour, **how fast is gas being consumed?**

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- $\dfrac{dG}{dt} = \dfrac{dG}{ds} \cdot \dfrac{ds}{dt} = 0.05 \cdot 30 = 1.5$ gallons/hour.

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- $df/du = 1/u$ and $du/dt = 3$.
- $df/dt = df/du \cdot du/dt = (\ln u) \cdot 3 = 3\ln(3t + 1)$, since $u = 3t + 1$.

Reminder

Theorem

$$\frac{d}{dx}(x^n) = nx^{n-1}, \quad \frac{d}{dx}(a^x) = (\ln a)a^x, \quad \frac{d}{dx}(\ln x) = \frac{1}{x}.$$

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(b) $y = 5(2x^2 + 4)^2$

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(c) $P = (3x + 1) + 4e^{9x+3}$

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- $\ln(q+4^q)$
- e^{-x^2+3x+1}

Relative Rate of change and logarithm

The relative rate of change of a function $z = f(t)$ to be

$$\frac{f'(t)}{f(t)} = \frac{1}{z} \cdot f'(t) = \frac{1}{z}\frac{dz}{dt}.$$

Since

$$\frac{d}{dt}(\ln z) = \frac{1}{z}\frac{dz}{dt},$$

we have

Theorem

For any positive function $f(t)$,

$$\textit{Relative rate of change of } f(t) = \frac{d}{dt}(\ln f(t)).$$

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- Relative rate of change of $f(t)$ is $\frac{d}{dt}(\ln z) = k$.

Examples

- The surface area S of a mammal, in cm^2, is a functions of the body mass, M, of the mammal, in kg, and is given by $S = 1095 \cdot M^{2/3}$. Find the relative rate of change of S with respective to M and evaluate for human with body mass 70 kg. Interpret the answer.

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- The surface area of a human with body mass 70 kg increases by about 0.95 % if the body mass increases by 1 kg.