Enumeration of Hybrid Domino-Lozenge Tilings

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Introduction

The field of exact enumeration of tilings dates back to the early 1900's when McMahon proved his theorem on the number of lozenge (rhombus) tilings of a hexagon with sides a, b, c, a, b, c on the triangular lattice. A large body of related work emerged in the last couple of decades, centered on families of lattice regions whose tilings are enumerated by simple product formulas.

In 1999, James Propp listed 20 open problems on the enumeration of tilings (updated with 12 addition open problems in [6]). Most of those problems have been proved in the meanwhile, but some are still open. We solve and generalize one of these open problems (Problem 16). Our method also provides a new proof and a generalization for a related result of Douglas [1].

TERMINOLOGY

- A **tile** is the union of two elementary regions that share an edge.
- A **tiling** of a region is a way to cover it by tiles so that there are no gaps or overlaps.
- Denote by **T(R)** the number of tilings of region R.

McMahon 1900. $T(H_{a,b,c}) = \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}$

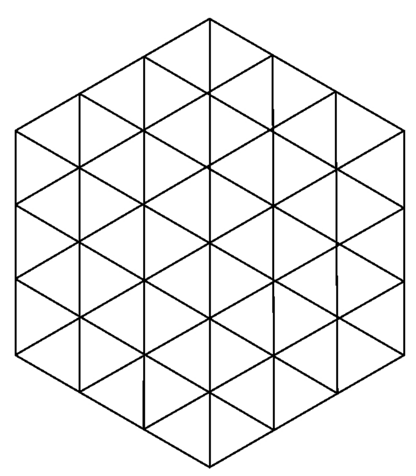

Figure 1. The region $H_{3,3,3}$ and a tiling

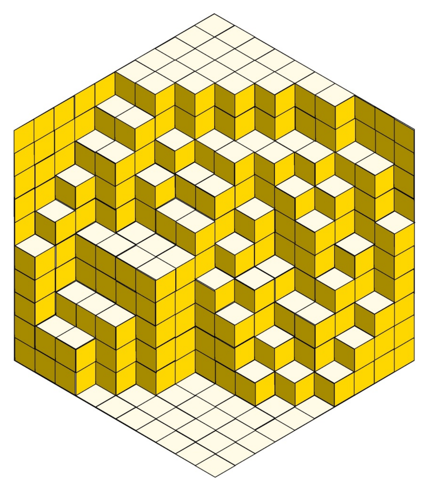

Figure 2. A tiling of $H_{10,10,10}$

Kasyeleyn-Temperley-Fisher 1961.

$$T(C_{2m,2n}) = 2^{2mn}\prod_{j=1}^{m}\prod_{k=1}^{n}\left(\cos^2\left(\frac{j\pi}{2m+1}\right)+\cos^2\left(\frac{k\pi}{2n+1}\right)\right)$$

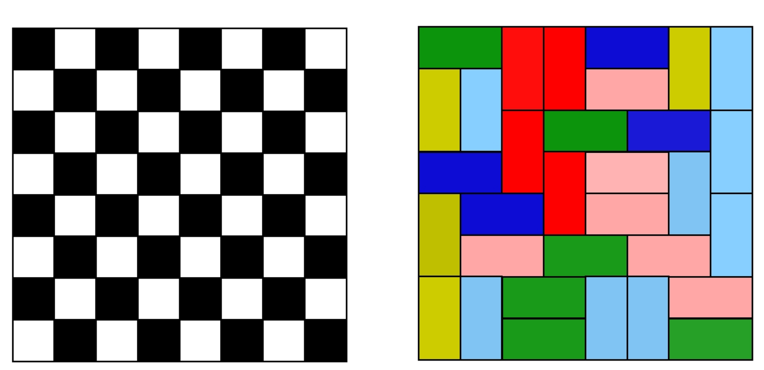

Figure 3. The 8 x 8 chess board and a tiling

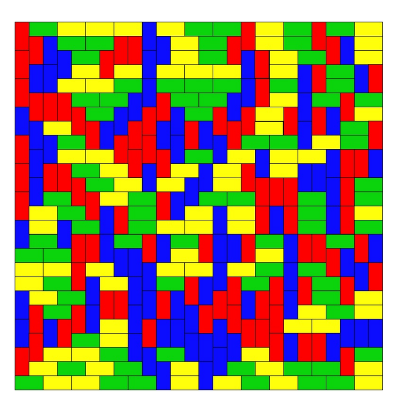

Figure 4. A tiling of 26 x 26 chess board

Elkies-Kuperberg-Larsen-Propp 1991. (Aztec Diamond Theorem)

$$T(AD_n) = 2^{n(n+1)/2}$$

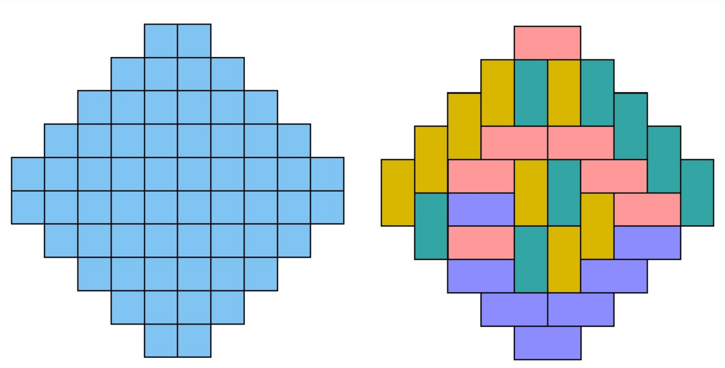

Figure 5. The Aztec diamond AD_5 and a tiling

Figure 6. An Aztec temple

Douglas 1996. $T(D_n) = 2^{2n(n+1)}$

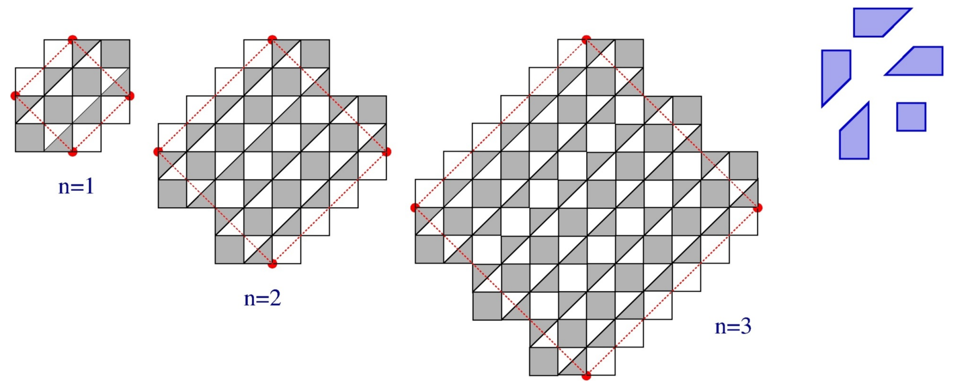

Figure 7. Douglas' regions D_1, D_2 and D_3; and five types of their tiles

Problem 16. Find a formula for the number of tilings of a quasi-hexagonal region on the square lattice with every third diagonal drawn in.

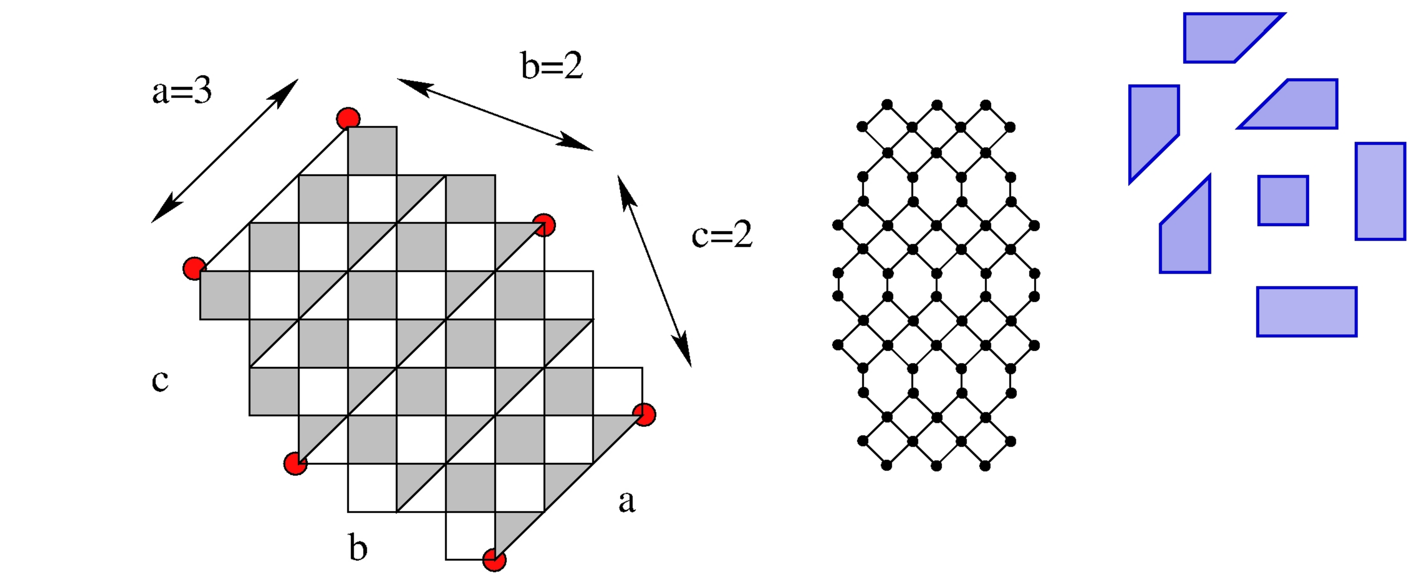

Figure 8. The quasi-hexagonal region of sides 3,2,2,3,2,2; its dual graph and seven types of tiles

Theorem 1 (T. L.)

$$T(H_a(d_1,\dots,d_k;\ d'_1,\dots,d'_l)) = 2^{C+C'-h(2q-h+1)}T(H_{h,q-h,h})$$

- a = |AF|, q=|BE|
- C = the number of black squares and black up-pointing triangles in the top part
- C'= the number of white squares and white down-pointing triangles in the bottom part
- h = the number of rows of black squares + the number of rows of black up-pointing triangles in the top part
  = the number of rows of white squares + the number of rows of white down-pointing triangles in the bottom part

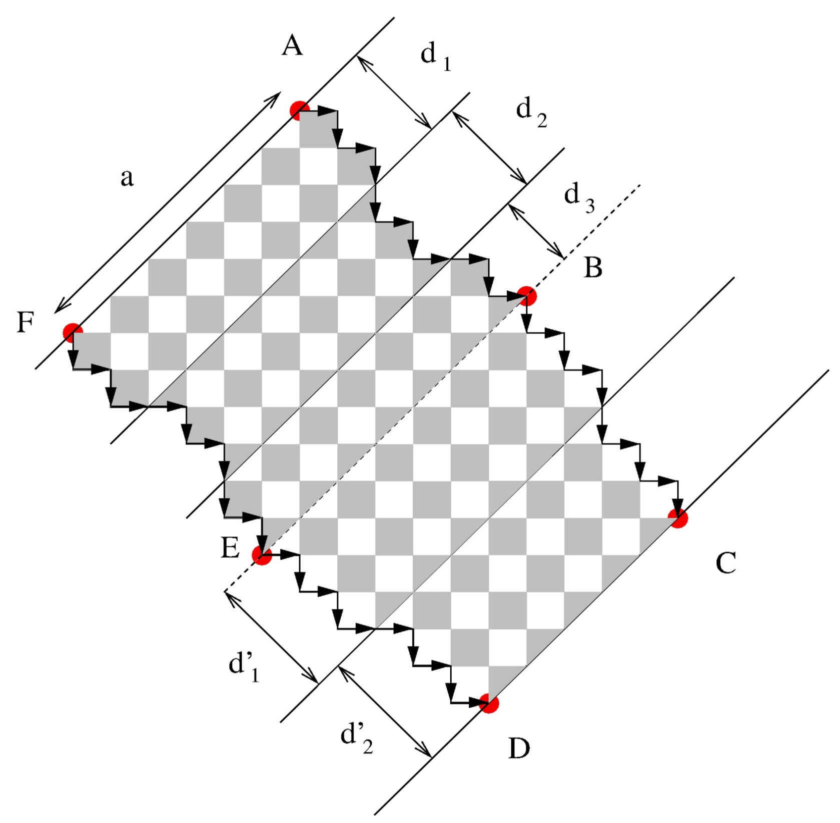

Figure 9. The region $H_6(4,4,3;\ 5,5)$

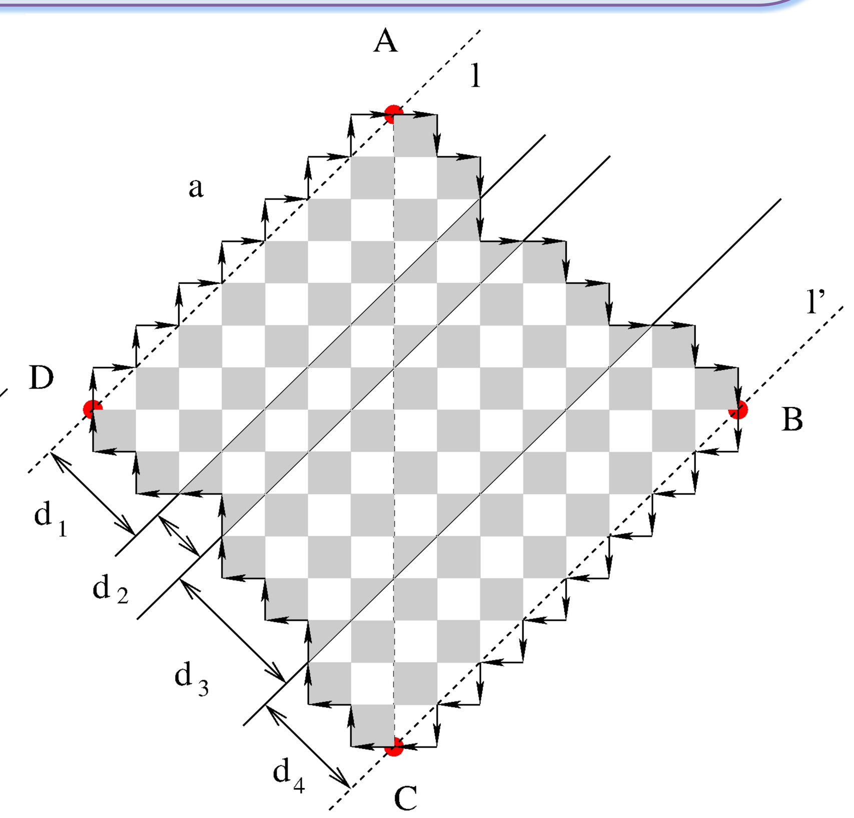

Figure 10. The region $D_7(4,2,5,4)$

Theorem 2 (T. L.)

$$T(D_a(d_1,\dots,d_k)) = 2^{C-h(h+1)/2}$$

- a=|AD|
- C= the number of black squares and black up-pointing triangles
- h=|BC|= the number of rows of black squares + the number of rows of black up-pointing triangles

Definitions

- The **dual graph** G of R is the graph whose vertices are the elementary regions, and whose edges connect two elementary regions precisely when they share an edge.
- A **perfect matching (or dimer covering)** of a graph G is a collection of edges such that each vertex of G is incident to precisely one edge in the collection

Method

- The number of tilings of the region R is equal to the number of perfect matchings of the dual graph G; denote the latter by **M(G)**.
- Use the **subgraph replacement method:** Replace a part of the dual graph by a new graph so that its number of perfect matchings changes by a simple factor.
- To prove Theorem 1 we apply repeatedly the replacement method until we get the dual graph of a hexagon, and then apply McMahon's formula.
- To prove Theorem 2 we apply repeatedly the replacement method until we get the dual graph of an Aztec diamond, and then apply the Aztec diamond theorem.

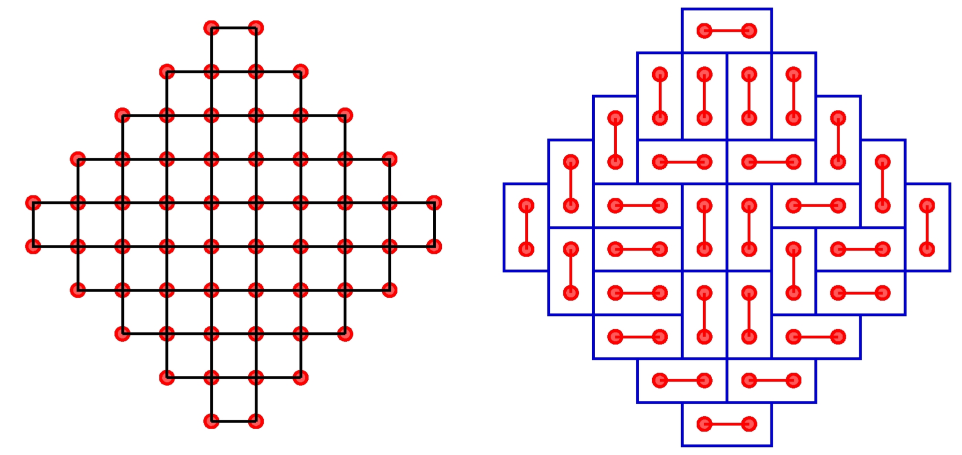

Figure 11. The equivalence between tilings and perfect matchings

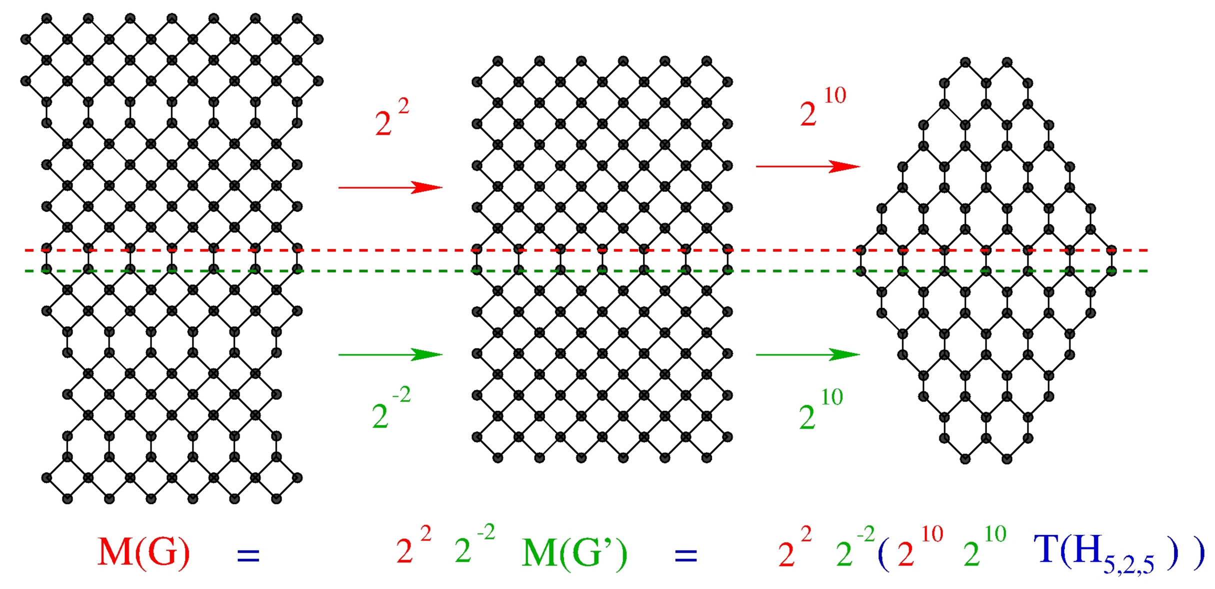

Figure 12. The subgraph replacements used in the proof of Theorem 1

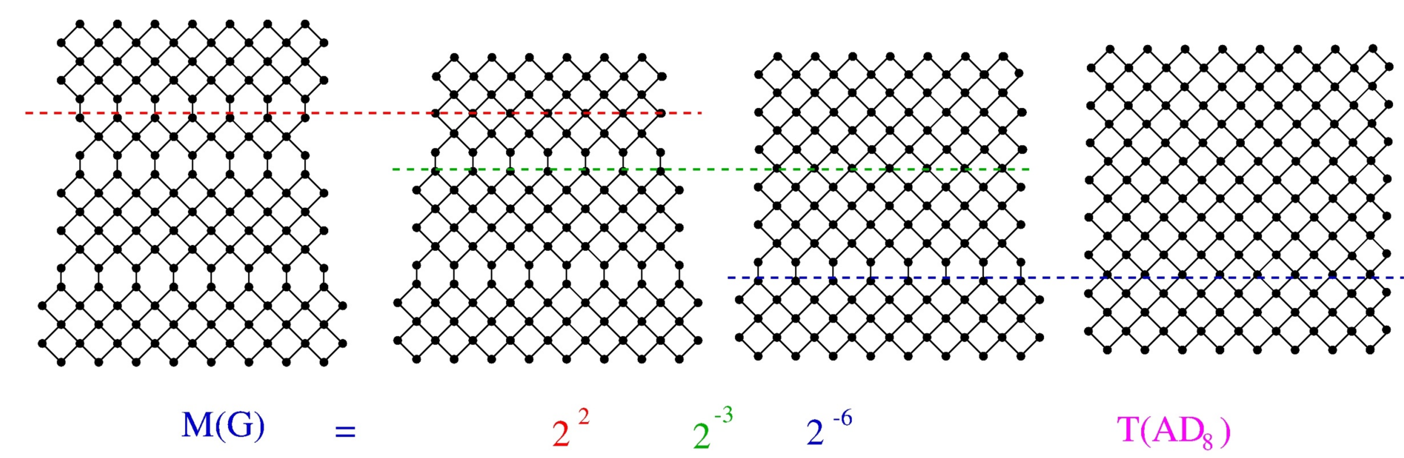

Figure 13. The subgraph replacements used in the proof of Theorem 2

References

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[2] N. Elkies et al., *Alternating-sign matrices and domino tilings,* J. Algeberaic Combin. **1** (1992), 111-132.

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[4] T. Lai, *Enumeration of hybrid domino-lozenge tilings*, submitted to J. Combin. Theory Ser. A

[5] P. A. MacMahon, *Combinatory Analysis*, vol. **2**, Cambridge University Press (1916)

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[7] H. N. V. Temperley and M. E. Fisher, *Dimer problem in statistical mechanics- an exact result*, Phil. Mag. **6** (1961), 1061-1063.

[8] http://wikipedia.org