Enumeration of domino tilings of a double Aztec rectangle

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Introduction

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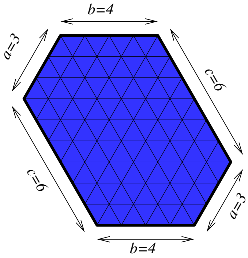
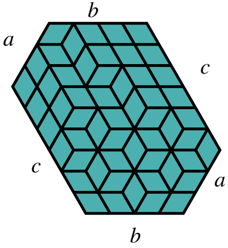
- They very are separated; there were not significant connections.
- Our main result is a such connection.

Lozenge tilings of a hexagon





- A lozenge (or unit rhombus) is the union of two adjacent unit equilateral triangles.
- A lozenge tiling of a region R on the triangular lattice is a covering of R by lozenges so that there are no gaps or overlaps.

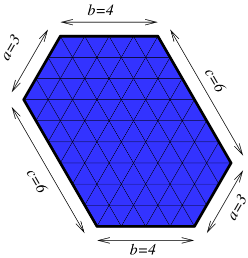
Semi-regular hexagons

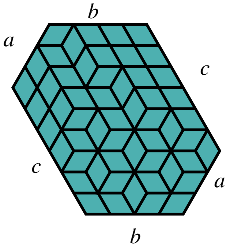
Theorem (MacMahon $\sim$ 1900)

$$\mathbb{T}(Hex(a,b,c)) = \frac{\mathbf{H}(a)\,\mathbf{H}(b)\,\mathbf{H}(c)\,\mathbf{H}(a+b+c)}{\mathbf{H}(a+b)\,\mathbf{H}(b+c)\,\mathbf{H}(c+a)},$$

where the "hyperfactorial" is defined as

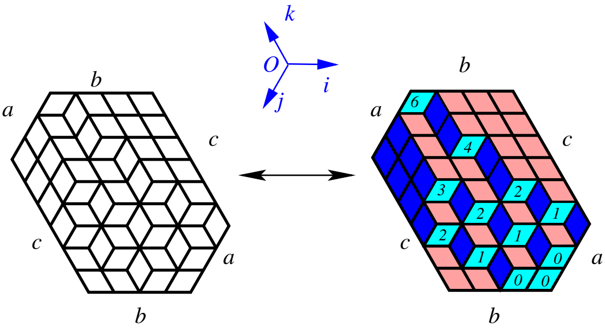
$$\mathbf{H}(n) := 0!\cdot 1!\cdot 2!\dots(n-1)!.$$





Lozenge tilings and plane partition

6	4	2	1
3	2	1	0
2	1	0	0



MacMahon's q-Theorem

Theorem (MacMahon's q-Theorem)

$$\sum_{\pi} q^{vol(\pi)} = \frac{\mathbf{H}_q(a)\,\mathbf{H}_q(b)\,\mathbf{H}_q(c)\,\mathbf{H}_q(a+b+c)}{\mathbf{H}_q(a+b)\,\mathbf{H}_q(b+c)\,\mathbf{H}_q(c+a)},$$

where the sum is taken over all monotonic stacks of unit cubes (plane partitions) π *fitting in an* $a \times b \times c$ *box.*

Definition:

- q-integer $[n]_q := 1 + q + q^2 + \ldots + q^{n-1}$
- q-factorial $[n]_q! = [1]_q[2]_q \ldots [n]_q$,
- q-hyperfactorial $\mathbf{H}_q(n) = [0]_q![1]_q! \ldots [n-1]_q!$.

Domino Tilings

Study of domino tilings came from statistical mechanics with the work of Kasteleyn, and Temperley and Fisher in 1961.

Theorem (Elkies, Kuperberg, Larsen and Propp 1992)

The Aztec diamond of order n has $2^{n(n+1)/2}$ domino tilings.

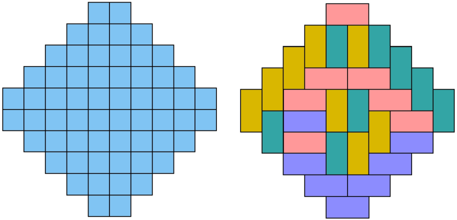
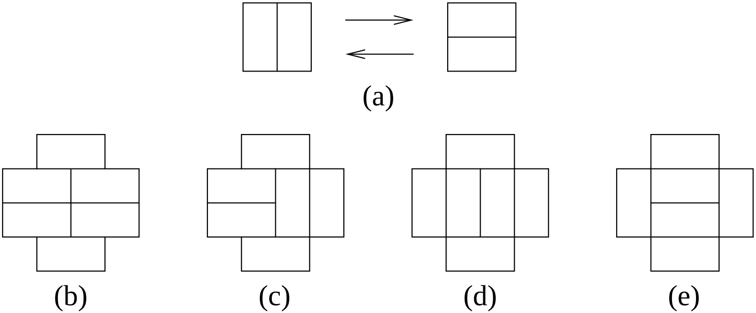


Figure: The Aztec diamond of order 5 and one of its tilings.

Rank of a domino tiling



- The all-horizontal domino T_0 has rank 0.
- The rank $r(T)$ of a tiling T is the smallest number of elementary moves to obtain the tiling T from T_0.

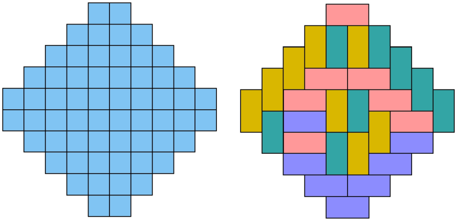
The Weighted Aztec Diamond Theorem

Theorem (Weighted Aztec Diamond Theorem)

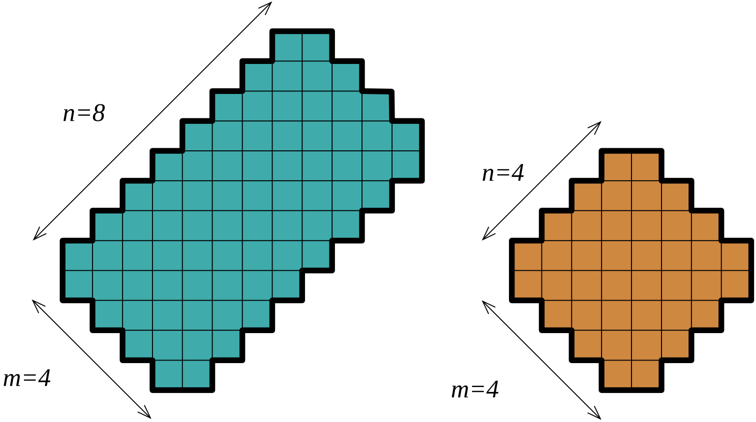
For any positive integer n and indeterminates t and q

$$\sum_{T} t^{v(T)} q^{r(T)} = \prod_{k=0}^{n-1} (1 + tq^{2k+1})^{n-k}, \qquad (1)$$

where $v(T)$ is half number of vertical dominoes in T.

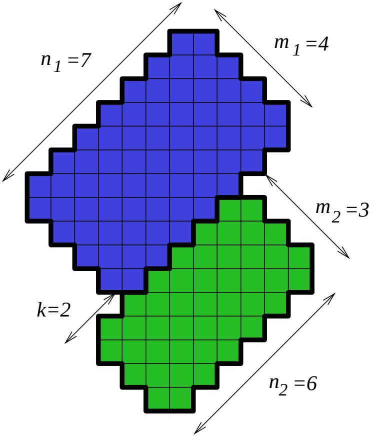


The Aztec Rectangle



(a) (b)

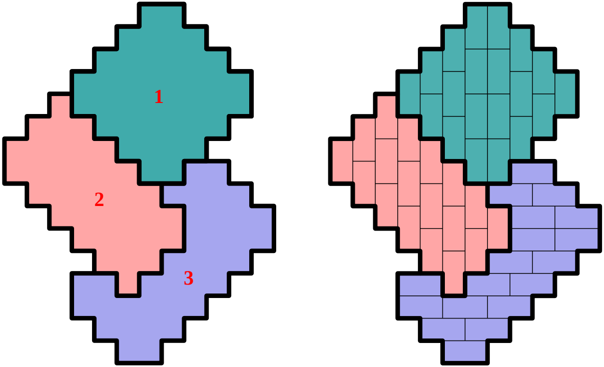
The Double Aztec Rectangle



$$\mathcal{DR}^{m_2,n_2}_{m_1,n_1,k}$$

Define the rank of a tiling

In general, $\mathcal{DR}^{m_2,n_2}_{m_1,n_1,k}$ does not have all-horizontal domino tiling.



(a) (b)

Main Theorem

Theorem (L. 2016)

Assume that $m_1 \le n_1$, $m_2 \le n_2$, $k \le \min(m_2, n_2 - 1)$, *and* $n_1 - m_1 = n_2 - m_2$. *Then*

$$\sum_{T \in \mathcal{T}(\mathcal{DR}^{m_2,n_2}_{m_1,n_1,k})} t^{v(T)} q^{r(T)} = t^{\binom{m_1+1}{2}+\binom{m_2+1}{2}+(n_1-m_1)(m_1+k)/2} q^{E}$$

$$\times \prod_{i=0}^{m_1-1} (1 + t^{-1} q^{2i+1})^{m_1-i} \prod_{i=0}^{m_2-1} (1 + t^{-1} q^{-2i-1})^{m_2-i}$$

$$\times P_{q^2}(n_1 - m_1, m_2 - k + 1, m_1 + k), \tag{2}$$

where

$$P_q(a,b,c) = \frac{\mathbf{H}_q(a)\,\mathbf{H}_q(b)\,\mathbf{H}_q(c)\,\mathbf{H}_q(a+b+c)}{\mathbf{H}_q(a+b)\,\mathbf{H}_q(b+c)\,\mathbf{H}_q(c+a)}.$$

A consequence

Corollary

Assume that $m_1 \leq n_1$, $m_2 \leq n_2$, $k \leq \min(m_2, n_2 - 1)$, *and* $n_1 - m_1 = n_2 - m_2$. *Then*

$$\mathbb{T}\left(\mathcal{DR}^{m_2,n_2}_{m_1,n_1,k}\right) = 2^{\binom{m_1+1}{2}+\binom{m_2+1}{2}} \times \mathbb{T}\left(Hex(n_1 - m_1, m_2 - k + 1, m_1 + k)\right).$$

Remark: The conditions $m_1 \leq n_1$, $m_2 \leq n_2$, $k \leq \min(m_2, n_2 - 1)$, and $n_1 - m_1 = n_2 - m_2$ to make sure the region has tilings.

A bijection between tilings and perfect matchings

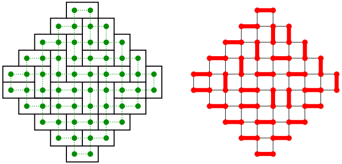


Figure: Bijection between tilings of the Aztec diamond of order 5 and perfect matchings of its dual graph.

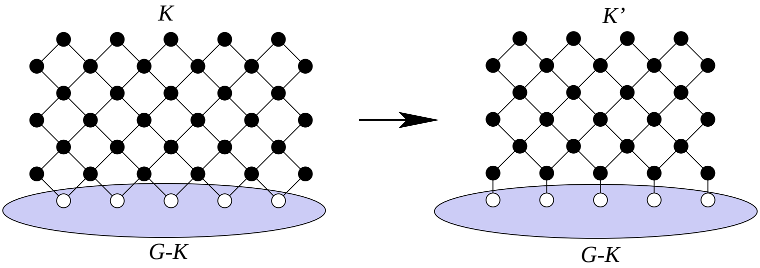
- The dual graph of a region R is the graph whose vertices are the "cells" in R and whose edges connect precisely two adjacent cells.
- A perfect matching of a graph G is a collection of disjoint edges covering all vertices of G.

Idea of the proof

Idea: "Transform" the dual graph of a double Aztec rectangle into the dual graph of a hexagon.

Subgraph replacement

$$\mathbf{M}(G) = 2^{\#\text{ rows of diamonds in } K}\mathbf{M}(G')$$



Compound replacement

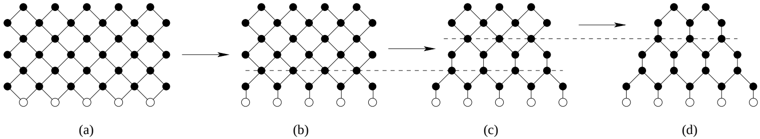
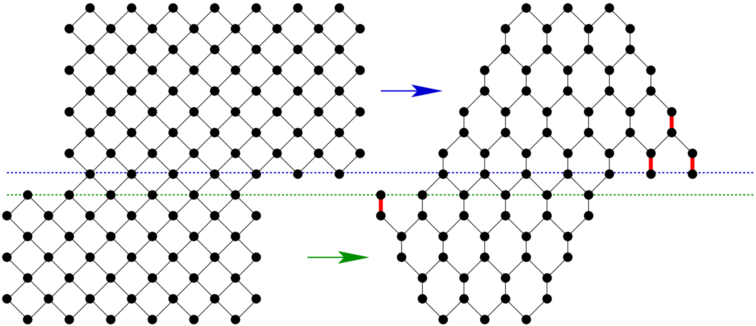
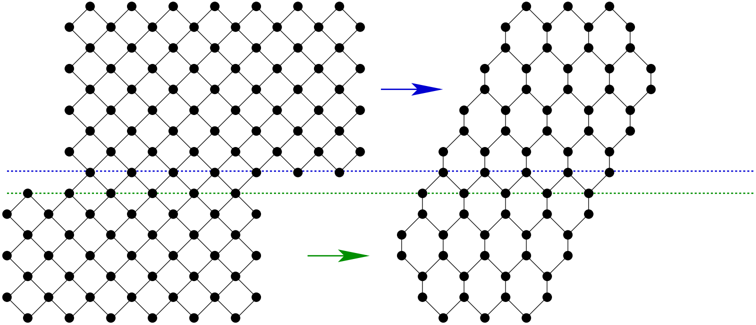


Figure: The compound replacement

Applying the compound replacement



Applying the subgraph replacement



$$\mathbb{T}\left(\mathcal{DR}^{m_2,n_2}_{m_1,n_1,k}\right) = 2^{\binom{m_1+1}{2}+\binom{m_2+1}{2}}$$

$$\times\,\mathbb{T}\left(Hex(n_1 - m_1, m_2 - k + 1, m_1 + k)\right).$$

The End

Thank You!