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Topics in  
Probability Theory and Stochastic Processes  
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Stirling’s Formula Derived from Elementary Sequences and Series

Rating

Mathematicians Only: prolonged scenes of intense rigor.
Section Starter Question

What is the geometric summation formula? How can you use the geometric sum formula to derive the series expansion for \( \log(1 + x) \)? What do you need to know about the geometric sum formula to justify its use to derive the series expansion for \( \log(1 + x) \)?

Key Concepts

1. Stirling’s Formula, also called Stirling’s Approximation, is the asymptotic relation

   \[
   n! \sim \sqrt{2\pi n} n^{n+1/2} e^{-n}.
   \]

2. The formula is useful in estimating large factorial values, but its main mathematical value is in limits involving factorials.

3. An improved inequality version of Stirling’s Formula is

   \[
   \sqrt{2\pi n} n^{n+1/2} e^{-n+1/(12n)} < n! < \sqrt{2\pi n} n^{n+1/2} e^{-n+1/(12n+1)}.
   \]

4. Some related inequalities and asymptotics for binomial coefficients are

   \[
   \binom{n}{k} \leq \binom{en}{k} \leq \left( \frac{en}{k} \right)^k
   \]

   and

   \[
   \binom{n}{k} \sim \left( \frac{n^k}{k!} \right)
   \]

   if \( k = o(n^{1/2}) \) as \( n \to \infty \).
Vocabulary

1. **Stirling’s Formula**, also called Stirling’s Approximation, is the asymptotic relation

   \[ n! \sim \sqrt{2\pi n^{n+1/2}e^{-n}}. \]

2. The “double factorial” notation is \( n!! = n \cdot (n - 2) \cdots 2 \) if \( n \) is even, and \( n!! = n \cdot (n - 2) \cdots 3 \cdot 1 \) if \( n \) is odd.

Mathematical Ideas

**Stirling’s Formula**, also called Stirling’s Approximation, is the asymptotic relation

\[ n! \sim \sqrt{2\pi n^{n+1/2}e^{-n}}. \]

The formula is useful in estimating large factorial values, but its main mathematical value is in limits involving factorials. Another attractive form of Stirling’s Formula is:

\[ n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n. \]

An improved inequality version of Stirling’s Formula is

\[ \sqrt{2\pi n^{n+1/2}e^{-n+1/(12n+1)}} < n! < \sqrt{2\pi n^{n+1/2}e^{-n+1/(12n)}}. \]  \hspace{1cm} (1)


A consequence of the improved inequality is the simple and useful inequality about Stirling’s Formula for all \( n \)

\[ \sqrt{2\pi n^{n+1/2}e^{-n}} < n!. \]

Here we rigorously derive Stirling’s Formula using elementary sequences and series expansions of the logarithm function, based on the sketch in Kazarinoff [2], based on the note by Nanjundiah [3].
A weak form of Stirling’s Formula

Theorem 1.
\[ \sqrt{n!} \sim \frac{n}{e} \]

Remark. This form of Stirling’s Formula is weaker than the usual form since it does not give direct estimates on \( n! \). On the other hand, it avoids the determination of the asymptotic constant \( \sqrt{2\pi} \) which usually requires Wallis’s Formula or equivalent. For many purposes of estimation or limit taking this version of Stirling’s Formula is enough, and the proof is elementary. The proof is taken from [5, pages 314-315].

Proof. 1. Start from the series expansion for the exponential function and then crudely estimate:
\[
e^n = 1 + \frac{n}{1!} + \cdots + \frac{n^{n-1}}{(n-1)!} + \frac{n^n}{n!} \left( 1 + \frac{n}{n+1} + \frac{n^2}{(n+1)(n+2)} + \cdots \right)
\[
< \frac{n^n}{n!} + \frac{n^n}{n!} \sum_{j=0}^{\infty} \left( \frac{n}{n+1} \right)^j
\]
\[
= (2n+1) \frac{n^n}{n!}.
\]

2. On the other hand, \( e^n > \frac{n^n}{n!} \) by dropping all but the \( \frac{n^n}{n!} \) term from the series expansion for the exponential.

3. Rearranging these two inequalities
\[
\frac{n^n}{e^n} < n! < \frac{(2n+1)n^n}{e^n}.
\]

4. Now take the \( n \)th root of each term, and use the fact that \( \sqrt{2n+1} \to 1 \) as \( n \to \infty \).

\[ \Box \]

Stirling’s Formula from sequences and series

Let \( a_n = \frac{n!}{n^{n+1/2}e^{-n}} \). Then \( \frac{a_n}{a_{n+1}} = \left( \frac{n+1}{n} \right)^{n+1/2} e^{-1} \) and \( \log \left( \frac{a_n}{a_{n+1}} \right) = (n + 1/2) \log(1 + 1/n) - 1 \).
Lemma 2. For $|x| < 1$, 

\[
\log\left(\frac{1 + x}{1 - x}\right) = 2 \sum_{k=0}^{\infty} \frac{1}{(2k + 1)x^{2k+1}}.
\]

Proof. Left as an exercise. \qed

Note then that

\[
\log\left(1 + \frac{1}{n}\right) = \log\left(1 - \frac{1}{2n+1}\right) = 2 \sum_{k=0}^{\infty} \frac{1}{(2k + 1)(2n + 1)^{2k+1}}.
\]

Then

\[
\log\left(\frac{a_n}{a_{n+1}}\right) = \left(\frac{2n+1}{2}\right)\left(2 \sum_{k=0}^{\infty} \frac{1}{(2k + 1)(2n + 1)^{2k+1}}\right) - 1
\]

\[
= \sum_{k=1}^{\infty} \frac{1}{(2k + 1)(2n + 1)^{2k}} = \sum_{k=0}^{\infty} \frac{1}{(2k + 3)(2n + 1)^{2k+2}}.
\]

Now coarsely estimating the denominators

\[
\log\left(\frac{a_n}{a_{n+1}}\right) \leq \frac{1}{2(2n + 1)^2} \sum_{k=0}^{\infty} \frac{1}{(2n + 1)^{2k}}.
\]

Lemma 3.

\[
\log(a_{n+1}) < \log(a_n) < \frac{1}{12n} - \frac{1}{12(n + 1)} + \log(a_{n+1})
\]

Proof. 1. Let $f(x) = (x + 1/2) \log(1 + 1/x) - 1$, so $\log\left(\frac{a_n}{a_{n+1}}\right) = f(n)$.

2. Observe that $f(x) \to 0$ as $x \to \infty$.

3. Also $f'(x) = \log(1 + 1/x) - \frac{2x+1}{2x^2+2x}$.

4. Because $f'(x) < 0$ for $x > 1$ (proof left as an exercise) $f(x)$ is decreasing from $(3/2) \log(2) - 1 \approx 0.0397$ to $0$ as $x$ increases.
5. Hence \(0 < \log \left( \frac{a_n}{a_{n+1}} \right)\).

6. By Lemma 2

\[
\log \left( \frac{a_n}{a_{n+1}} \right) \leq \frac{1}{3(2n+1)^2} \sum_{k=0}^{\infty} \frac{1}{(2n+1)^{2k}}.
\]

7. The sum is a geometric sum, so

\[
0 < \log \left( \frac{a_n}{a_{n+1}} \right) < \frac{1}{3(2n+1)^2} \frac{1}{1 - (1/(2n+1)^2)} = \frac{1}{12n(n+1)}.
\]

8. Expand \(1/(12n(n+1))\) in partial fractions and add \(\log(a_{n+1})\) throughout to get

\[
\log(a_{n+1}) < \log(a_n) < \frac{1}{12n} - \frac{1}{12(n+1)} + \log(a_{n+1}).
\]

\[\square\]

Define \(x_n = \log(a_n) - \frac{1}{12n}\), so Lemma 3 shows that \(x_n\) is an increasing sequence, \(x_n < x_{n+1}\). That is,

\[
x_n = \log(a_n) - \frac{1}{12n} < \log(a_{n+1}) - \frac{1}{12(n+1)} = x_{n+1}. \tag{2}
\]

Define \(y_n = \log(a_n)\), and then the left-side inequality in Lemma 3 shows that \(y_n\) is a decreasing sequence that is, \(y_{n+1} < y_n\). By the definition of \(x_n\) and \(y_n\), \(x_n < y_n\) and \(|x_n - y_n| = 1/(12n)\), so \(|x_n - y_n| \to 0\) as \(n \to \infty\). Therefore \(\sup x_n = \inf y_n\) and call the common value \(\lambda\). By continuity, \(\lim_{n \to \infty} a_n = e^\lambda\).

Using elementary properties of limits

\[
e^\lambda = \lim_{n \to \infty} a_n = \left( \lim_{n \to \infty} a_n \right)^2 = \lim_{n \to \infty} \frac{(a_n)^2}{a_{2n}}. \tag{3}
\]

However,

\[
\frac{(a_n)^2}{a_{2n}} = \sqrt{\frac{2^n}{n}} \frac{2 \cdot 4 \ldots (2n-2) \cdot 2n}{1 \cdot 3 \ldots (2n-3) \cdot (2n-1)}. \tag{4}
\]

The demonstration is left as an exercise.
Lemma 4 (Wallis’ Formula).

\[
\lim_{n \to \infty} \frac{(2n) \cdot (2n) \ldots 2 \cdot 2}{(2n+1) \cdot (2n-1) \cdot (2n-3) \ldots 3 \cdot 3 \cdot 1} = \frac{\pi}{2}.
\]

Proof. See the proofs in Wallis Formula.

Using the continuity of the square root function

\[
\lim_{n \to \infty} \left( \sqrt{\frac{1}{2n+1}} \right) \frac{(2n) \cdot (2n-2) \ldots 4 \cdot 2}{(2n-1) \cdot (2n-3) \ldots 5 \cdot 3 \cdot 1} = \frac{\sqrt{\pi}}{2}.
\]

Now multiplying both sides by 2 and rewriting the leading square root sequence, get

\[
\lim_{n \to \infty} \left( \sqrt{\frac{2}{n}} \frac{2n}{2n+1} \right) \frac{(2n) \cdot (2n-2) \ldots 4 \cdot 2}{(2n-1) \cdot (2n-3) \ldots 5 \cdot 3 \cdot 1} = \sqrt{2\pi}.
\]

Then since

\[
\lim_{n \to \infty} \sqrt{\frac{2n}{2n+1}} = 1
\]

equation [4] is

\[
e^\lambda = \lim_{n \to \infty} a_n = \sqrt{\frac{2}{n}} \frac{2 \cdot 4 \ldots (2n-2) \cdot 2n}{1 \cdot 3 \ldots (2n-3) \cdot (2n-1)} = \sqrt{2\pi}.
\]

Equivalently, unwrapping the definition of \(a_n = \frac{n!}{n^{n+1/2}e^{-n}}\) this is exactly Stirling’s Formula

\[
n! \sim \sqrt{2\pi n^{n+1/2}e^{-n}}.
\]

Using the definitions \(x_n = \log(a_n) - \frac{1}{12n}\) and \(y_n = \log(a_n)\), the inequality \(x_n < y_n\), and the least upper bound and greatest lower bound limit in equation [3] we can express Stirling’s Formula in inequality form

\[
\sqrt{2\pi n^{n+1/2}e^{-n}} < n! < \sqrt{2\pi n^{n+1/2}e^{-n+1/(12n)}}.
\]

This is almost as good as the inequality [1]. This also gives a proof of the simple and useful inequality about Stirling’s Formula, for all \(n\)

\[
\sqrt{2\pi n^{n+1/2}e^{-n}} < n!.
\]

Remark. This proof of Stirling’s Formula and the inequality [1] is the easiest, the shortest and the most elementary of the Stirling’s Formula proofs. These are all definite advantages. The main disadvantage of this proof is that it requires the form of Stirling’s Formula before starting, in order to create the sequence \(a_n\) which is the main object of the proof.
Figure 1: Example of the lemma with $\alpha = 1$ (green) and $\alpha = 0$ (red).

**Stirling’s Formula from Wallis’ Formula**

**Lemma 5.** If $\alpha \in \mathbb{R}$, then the sequence

$$a_\alpha(n) = \left(1 + \frac{1}{n}\right)^{n+\alpha}$$

is decreasing if $\alpha \in \left[\frac{1}{2}, \infty\right)$, and increasing for $n \geq N(\alpha)$ if $\alpha \in (-\infty, \frac{1}{2})$.

**Remark.** This lemma is based on a remark due to I. Schur, see [1], problem 168, on page 38 with solution on page 215.

**Proof.** 1. The derivative of the function $f(x) = \left(1 + \frac{1}{x}\right)^{x+\alpha}$ (defined on $[1, \infty)$) is

$$f'(x) = \left(1 + \frac{1}{x}\right)^{x+\alpha} \left(\ln\left(1 + \frac{1}{x}\right) - \frac{x + \alpha}{x(x+1)}\right).$$
2. Let
\[ g(x) = \left( \ln \left( 1 + \frac{1}{x} \right) - \frac{x + \alpha}{x(x + 1)} \right) \]
then
\[ g'(x) = \frac{(2\alpha - 1)x + \alpha}{x^2(x + 1)^2} \]
and \( \lim_{x \to \infty} g(x) = 0. \)

3. It follows that \( g(x) < 0 \) and so \( f'(x) < 0 \) when \( \alpha \geq 1/2 \) and \( x \geq 1 \), and \( f'(x) > 0 \) when \( \alpha < 1/2 \) and \( x \geq \max(1, \frac{\alpha}{1-2\alpha}) \). The monotonicity of \( a_\alpha(n) \) follows.

\[ \square \]

From the lemma, for every \( \alpha \in (0, 1/2) \) there is a positive integer \( N(\alpha) \) such that
\[ \left( 1 + \frac{1}{k} \right)^{k+\alpha} < e < \left( 1 + \frac{1}{k} \right)^{k+1/2} \]
for all \( k \geq N(\alpha) \). As a consequence, we get
\[ \prod_{k=n}^{2n-1} \left( 1 + \frac{1}{k} \right)^{k+\alpha} < e^n < \prod_{k=n}^{2n-1} \left( 1 + \frac{1}{k} \right)^{k+1/2} . \]

Rearrange the products with telescoping cancellations, using the upper
bound on the right as an example.

\[
\prod_{k=n}^{2n-1} \left(1 + \frac{1}{k}\right)^{k+1/2} = \left(1 + \frac{1}{n}\right)^{n+1/2} \left(1 + \frac{1}{n+1}\right)^{n+1/2} \cdots \left(1 + \frac{1}{2n-1}\right)^{2n-1+1/2}
\]

\[
= \left(\frac{n+1}{n}\right)^{n+1/2} \left(\frac{n+2}{n+1}\right)^{n+1/2} \cdots \left(\frac{2n}{2n-1}\right)^{2n-1+1/2}
\]

\[
= \left(\frac{2n}{n}\right)^{1/2} \left(\frac{2n+1}{2n-1}\right)^{1/2} \cdots \left(\frac{n+1}{n}\right)^{n} \left(\frac{n+2}{n+1}\right)^{n+1} \cdots \left(\frac{2n}{2n-1}\right)^{2n-1}
\]

\[
= \left(\frac{2n}{n}\right)^{1/2} \cdot \frac{(2n)^{2n-1}}{n^n \cdot (n+1) \cdots (2n-1)}
\]

\[
= 2^{1/2} \cdot \frac{(2n)^{2n-1}}{n^n \cdot (n+1) \cdots (2n-1)}
\]

Multiply by the last fraction by \(n!/n^n\) and Write more compactly,

\[
\frac{n!}{n^n} \cdot 2^{1/2} \cdot \left(\frac{2^{2n-1} \cdot n^{n-1}}{n+1) \cdots (2n-1)}\right)
\]

\[
= 2^{1/2} \cdot \frac{2^{n-1} \cdot 2^n \cdot n!}{n \cdot (n+1) \cdots (2n-1)}
\]

\[
= 2^{1/2} \cdot \frac{(2n)!!}{(2n-1)!!}
\]

This uses the “double factorial” notation \(n!! = n \cdot (n - 2) \cdots 2\) if \(n\) is even, and \(n!! = n \cdot (n - 2) \cdots 3 \cdot 1\) if \(n\) is odd. The lower bound product on the left is similar.

That is, after multiplying through by \(n!/n^{n+1/2}\),

\[
\frac{2^\alpha}{\sqrt{n}} \cdot \frac{(2n)!!}{(2n-1)!!} < \frac{n!e^n}{n^{n+1/2}} < \frac{2^{1/2} \cdot (2n)!!}{\sqrt{n}} \cdot \frac{(2n-1)!!}{(2n-1)!!}
\]

for all \(n \geq N(\alpha)\). Using the Wallis Formula,

\[
2^\alpha \sqrt{\pi} \leq \lim \inf_{n \to \infty} \frac{n!e^n}{n^{n+1/2}} \leq \lim \sup_{n \to \infty} \frac{n!e^n}{n^{n+1/2}} \leq \sqrt{2\pi}.
\]
Stirling’s formula follows by passing to the limit as $\alpha \to 1/2$.

**Remark.** This proof can be extended to an asymptotic formula for the Gamma function using log-convexity of the Gamma function. See [1].

### Related Asymptotic Formulas for Binomial Coefficients

**Theorem 6.**

\[
\left( \frac{n}{k} \right)^k \leq \binom{n}{k} \leq \left( \frac{en}{k} \right)^k
\]

**Proof.**

1. Start with

\[
\binom{n}{k} = \frac{n(n-1) \cdots (n-k+2)(n-k+1)}{k(k-1) \cdots 2 \cdot 1}.
\]

2. Since

\[
\frac{n}{k} \leq \frac{n-i}{k-i}
\]

for all $i = 0, 1, \ldots k-1$, the left inequality is immediate.

3. Since

\[
\binom{n}{k} = \frac{n(n-1) \cdots (n-k+2)(n-k+1)}{k(k-1) \cdots 2 \cdot 1} \leq \frac{n^k}{k!}
\]

and by a step in the proof of Theorem [1]

\[
\frac{1}{k!} \leq \left( \frac{e}{k} \right)^k
\]

the right inequality follows immediately.

**Theorem 7.**

\[
\binom{n}{k} \sim \left( \frac{n^k}{k!} \right)
\]

if $k = o(n^{1/2})$ as $n \to \infty$.

**Proof.**

1. The statement of the theorem is equivalent to showing

\[
\lim_{n \to \infty} \frac{n(n-1) \cdots (n-k+1)}{n^k} = \lim_{n \to \infty} \frac{k-1}{\prod_{j=1}^{k-1} (1 - j/n)} = 1
\]

if $k = o(n^{1/2})$.  

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2. In turn, this is equivalent to showing
\[ \lim_{n \to \infty} \log \left( \prod_{j=1}^{k-1} (1 - j/n) \right) = \lim_{n \to \infty} \sum_{j=1}^{k-1} \log (1 - j/n) = 0. \]

3. Using \( \log(1 - x) = x + O(x^2) \), the sum is
\[ \sum_{j=1}^{k-1} \log (1 - j/n) = \sum_{j=1}^{k-1} \left( \frac{j}{n} + O\left(\frac{j^2}{n^2}\right) \right) = \frac{k(k-1)}{n} + \frac{1}{n^2}O(k^3). \]

4. Now the hypothesis that \( k = o(n^{1/2}) \) comes into play so that \( \lim_{n \to \infty} \frac{k(k-1)}{n} = 0 \) and \( \lim_{n \to \infty} \frac{1}{n^2}O(k^3) = 0 \) establishing the desired limit.

\[ \square \]

Sources

The weak form of Stirling’s Formula is taken from [5]. The first sequence proof is adapted from the sketch in Kazarinoff [2] based on the note by Nanjundiah [3]. The second sequence proof using derivatives and monotonicity is adapted from [1]. The binomial coefficient limits are from lecture notes by Xavier Perez Gimenez.

Problems to Work for Understanding

1. Show that \( \sqrt{2n+1} \to 1 \) as \( n \to \infty \).

2. Show that for \( |x| < 1 \),
\[ \log \left( \frac{1+x}{1-x} \right) = 2 \sum_{k=0}^{\infty} \frac{1}{(2k+1)x^{2k+1}}. \]
3. Let \( f(x) = (x + 1/2) \log(1 + 1/x) - 1 \). Show that \( f(x) \) decreases to 0 as \( x \to \infty \).

4. Show that
\[
\frac{(a_n)^2}{a_{2n}} = \sqrt{\left(\frac{2}{n}\right) \frac{2 \cdot 4 \cdots (2n-2) \cdot 2n}{1 \cdot 3 \cdots (2n-3) \cdot (2n-1)}}.
\]

5. Show that the derivative of the function \( f(x) = (1 + \frac{1}{x})^{x+\alpha} \) (defined on \([1, \infty)\)) is
\[
f'(x) = \left(1 + \frac{1}{x}\right)^{x+\alpha} \left( \ln \left(1 + \frac{1}{x}\right) - \frac{x + \alpha}{x(x+1)} \right).
\]

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**Reading Suggestion:**

**References**


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