Topics in
Probability Theory and Stochastic Processes
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Stirling’s Formula Derived from the Gamma Function

Rating
Mathematicians Only: prolonged scenes of intense rigor.
Section Starter Question

Do you know of a function defined for, say the positive real numbers, which has the value \( n! \) for positive integers \( n \)?

Key Concepts

1. **Stirling’s Formula**, also called Stirling’s Approximation, is the asymptotic relation
   \[
   n! \sim \sqrt{2\pi n^{n+1/2}}e^{-n}.
   \]

2. The formula is useful in estimating large factorial values, but its main mathematical value is in limits involving factorials.

3. The **Gamma function** is
   \[
   \Gamma(z) = \int_0^\infty x^{z-1}e^{-x} \, dx.
   \]
   For an integer \( n \), \( \Gamma(n) = (n - 1)! \).

4. The Gamma function also has the asymptotic relation
   \[
   \lim_{t \to \infty} \frac{\Gamma(t + 1)e^t}{\sqrt{2\pi t^{t+1/2}}} = 1.
   \]
Vocabulary

1. **Stirling’s Formula**, also called Stirling’s Approximation, is the asymptotic relation

\[ n! \sim \sqrt{2\pi n^{n+1/2}e^{-n}}. \]

2. The **Gamma function** is

\[ \Gamma(z) = \int_{0}^{\infty} x^{z-1}e^{-x} \, dx. \]

For an integer \( n \), \( \Gamma(n) = (n - 1)! \).

Mathematical Ideas

**Stirling’s Formula**, also called Stirling’s Approximation, is the asymptotic relation

\[ n! \sim \sqrt{2\pi n^{n+1/2}e^{-n}}. \]

Another attractive form of Stirling’s Formula is:

\[ n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n. \]

The formula is sometimes useful for estimating large factorial values, but its main mathematical value is for limits involving factorials.

An improved inequality version of Stirling’s Formula is

\[ \sqrt{2\pi n^{n+1/2}e^{-n+1/(12n+1)}} < n! < \sqrt{2\pi n^{n+1/2}e^{-n+1/(12n)}}. \]


This article first formally defines the Gamma function and its most important property. A heuristic argument for Stirling’s formula for the Gamma function using asymptotics of integrals follows, based on some notes by P. Garrett, [1]. Finally, the article rigorously derives Stirling’s Formula using the Gamma function and estimates of the logarithm function, based on the short note by R. Michel, [2].
Definition of the Gamma Function

Lemma 1 (Factorial Integral).

\[ n! = \int_0^\infty x^n e^{-x} \, dx. \]

Proof. The proof is by induction. Start with \( n = 0, \)
\[ \int_0^\infty x^0 e^{-x} \, dx = \int_0^\infty e^{-x} \, dx = 1 = 0!. \]

Then in general, integrating by parts
\[ \int_0^\infty x^n e^{-x} \, dx = n \int_0^\infty x^{n-1} e^{-x} \, dx. \]
Inductively,
\[ n! = \int_0^\infty x^n e^{-x} \, dx. \]

Remark. The Gamma function is
\[ \Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \, dx. \]

Then the same integration by parts shows \( \Gamma(z) = (z-1)\Gamma(z-1) \) for \( z > 0. \)
For an integer \( n, \) Lemma 1 shows that \( \Gamma(n) = (n-1)! \) so the Gamma function
is an extension of the factorial to the positive real numbers.

Heuristic Derivation of Stirling’s Formula from Asymptotics of Integrals

By definition
\[ \Gamma(z+1) = \int_0^\infty x^z e^{-x} \, dx = \int_0^\infty e^{-x+z \log x} \, dx. \]
Note that \( -x + z \log x \) has a unique critical point at \( x = z. \) The critical
point is a maximum since the second derivative has value \( -1/z < 0 \) there.
The idea is to replace \(-x + z \log x\) by its second degree Taylor polynomial based at \(z\), then evaluate the resulting integral. Note that the second degree Taylor polynomial is a quadratic polynomial locally approximating a function. The idea is that most of the contribution to the integral comes from the maximum, and because of the negative exponential, contribution away from the maximum is slight.

So expand \(-x + z \log x\) in a second-degree Taylor polynomial at the critical point \(z\):

\[
-x + z \log x \approx -z + z \log z - \frac{1}{2!} z (x - z)^2.
\]

Approximate the Gamma function with

\[
\Gamma(z + 1) = z \Gamma(z) = \int_0^\infty e^{-x + z \log x} \, dx
\]

\[
\approx \int_0^\infty e^{-z + z \log z - \frac{x}{2}} (x - z)^2 \, dx
\]

\[
= e^{-z} z^z \int_0^\infty e^{-(x-z)^2/2} \, dx
\]

\[
\approx e^{-z} z^z \int_{-\infty}^\infty e^{-(x-z)^2/2} \, dx.
\]

Note that last evaluation of the integral is over the whole real line. By Lemma 4 of DeMoivre-Laplace Central Limit Theorem

\[
\int_{-\infty}^0 e^{-(x-z)^2/2} \, dx \leq \frac{1}{2z} e^{-z/4}.
\]

This is very small for large \(z\), so heuristically the integral over \((-\infty, \infty)\) is about the same as the integral over \([0, \infty)\).

To simplify the integral, replace \(x\) by \(zu\). Notice that a factor of \(z\) cancels from both sides of equation (2) giving

\[
\Gamma(z) = e^{-z} z^z \int_{-\infty}^\infty e^{-z(u-1)^2/2} \, du.
\]

Make another change of variables \(x = \sqrt{z}(u - 1)\) so that

\[
e^{-z} z^z \int_{-\infty}^\infty e^{-z(u-1)^2/2} \, du = e^{-z} z^z \sqrt{\frac{1}{2\pi}} \int_{-\infty}^\infty e^{-\frac{x^2}{2}} \, dx = e^{-z} z^z \sqrt{\frac{2\pi}{\sqrt{z}}}.
\]
The last equality is derived in Evaluation of the Gaussian Density Integral. Simplifying,

\[ \Gamma(z) \approx \sqrt{2\pi} e^{-z^2} z^{z-1/2}. \]

As an alternative, apply Laplace’s method (for asymptotics of integrals) to the integral in (2) and derive the same conclusion somewhat more rigorously.

**Rigorous Derivation of Stirling’s Formula**

**Lemma 2.**

\[ n! = n^n \sqrt{n} e^{-n} \int_{-\infty}^{\infty} g_n(y) \, dy \]

where

\[ g_n(y) = \left( 1 + \frac{y}{\sqrt{n}} \right)^n e^{-y\sqrt{n}} 1_{(-\sqrt{n}, \infty)}(y). \]

**Proof.** In Lemma 1 make the substitution \( x = \sqrt{n}y + n \) (or equivalently \( y = \frac{x}{\sqrt{n}} - \sqrt{n} \)) with \( dx = \sqrt{n} \, dy \) to give

\[
\int_0^{\infty} x^n e^{-x} \, dx = \int_{-\sqrt{n}}^{\infty} (y\sqrt{n} + n)^n e^{-(y\sqrt{n}+n)} \sqrt{n} \, dy \\
= n^n \sqrt{n} e^{-n} \int_{-\sqrt{n}}^{\infty} (y/\sqrt{n} + 1)^n e^{-(y\sqrt{n})} \, dy \\
= n^n \sqrt{n} e^{-n} \int_{-\infty}^{\infty} g_n(y) \, dy.
\]

**Remark.** This means that what remains is to show the integral \( \int_{-\infty}^{\infty} g_n(y) \, dy \) approaches the asymptotic constant \( \sqrt{2\pi} \).

**Lemma 3.** For \( |x| < 1 \),

\[
\left| \log(1 + x) - x + \frac{1}{2} x^2 \right| \leq \frac{1}{3} \frac{|x|^3}{1 - |x|}.
\]
Remark. Figure 1 is an illustration of the inequality.

Remark. Note further that
\[ \frac{|x|^3}{1 - |x|} \leq |x|^2 \quad (3) \]
for \(|x| \leq 1/2\). This will be used later to simplify the expression on the right side of the inequality in Lemma 3.

Proof. Expanding \(\log(1 + x)\) in a Taylor series and applying the triangle inequality
\[ \left| \log(1 + x) - x + \frac{1}{2} x^2 \right| \leq \sum_{k=3}^{\infty} \frac{|x|^3}{k}. \]
Grossly overestimating each term by using the least denominator and summing as a geometric series
\[ \sum_{k=3}^{\infty} \frac{|x|^3}{k} \leq \frac{1}{3} \frac{|x|^3}{1 - |x|}. \]
Lemma 4. 
\[ |e^a - e^b| \leq e^b \cdot |a - b| \cdot e^{a-b}. \]

Proof.
\[ |e^a - e^b| = e^b |e^{a-b} - 1| \]
Notice that \( e^x - 1 = \sum_{k=1}^{\infty} \frac{x^k}{k!} = x \sum_{k=1}^{\infty} \frac{x^{k-1}}{k!} \leq x \sum_{k=0}^{\infty} \frac{x^k}{k!} = xe^x. \) Alternatively let \( f(x) = e^x - 1 \) and \( g(x) = xe^x. \) Then \( f(0) = 0 = g(0) \), and \( f'(x) = e^x \leq e^x + xe^x = g'(x) \) for \( x > 0. \) Likewise \( f'(x) = e^x \geq e^x + xe^x = g'(x) \) for \( x < 0. \) Therefore \( e^x - 1 = f(x) \leq g(x) = xe^x \) for all \( x. \) Then \( e^b|e^{a-b} - 1| \leq e^b \cdot |a - b| \cdot e^{a-b}. \)

\[ \]

Lemma 5. For \( |y| \leq \frac{\sqrt{n}}{2} \)
\[ |g_n(y) - e^{-y^2/2}| \leq \frac{|y|^3}{\sqrt{n}} e^{-y^2/6}. \]

Proof. For \( |y| \leq \frac{\sqrt{n}}{2} \)
\[ |g_n(y) - e^{-y^2/2}| = \left| \left( 1 + \frac{y}{\sqrt{n}} \right)^n \right| \cdot e^{-y\sqrt{n}} 1_{(-\sqrt{n}, \infty)}(y) - e^{-y^2/2} \]
\[ = \left| e^{\log \left( \left( 1 + \frac{y}{\sqrt{n}} \right)^n \right) - y\sqrt{n}} 1_{(-\sqrt{n}, \infty)}(y) - e^{-y^2/2} \right|. \]
Using \( |y| \leq \frac{\sqrt{n}}{2} \)
\[ = \left| e^{\log \left( \left( 1 + \frac{y}{\sqrt{n}} \right)^n \right) - y\sqrt{n} - e^{-y^2/2} \right|. \]
Using Lemma 4
\[ \leq e^{-y^2/2} \cdot \left| \log \left( \left( 1 + \frac{y}{\sqrt{n}} \right)^n \right) - y\sqrt{n} + y^2/2 \right| \cdot e^{\left| \log \left( \left( 1 + \frac{y}{\sqrt{n}} \right)^n \right) - y\sqrt{n} + y^2/2 \right|}, \]
then with rules of logarithms
\[ = e^{-y^2/2} \cdot n \cdot \left| \log \left( 1 + \frac{y}{\sqrt{n}} \right) - \frac{y}{\sqrt{n}} + \frac{y^2}{2n} \right| \cdot e^{n \left| \log \left( \left( 1 + \frac{y}{\sqrt{n}} \right)^n \right) - \frac{y}{\sqrt{n}} + \frac{y^2}{2n} \right|}, \]
and using Lemma 3
\[ \leq e^{-y^2/2} \cdot n \cdot \frac{1}{3} \frac{|y|^3/n^{3/2}}{1 - |y|/\sqrt{n}} e^{n^{1/2} \frac{|y|^3/n^{3/2}}{1 - |y|/\sqrt{n}}} \]

Making a coarse estimate on the fraction
\[ \leq e^{-y^2/2} \cdot n \cdot \frac{1}{3} \frac{|y|^3/n^{3/2}}{1 - (\sqrt{n}/2)/\sqrt{n}} e^{n^{1/2} \frac{|y|^3/n^{3/2}}{1 - |y|/\sqrt{n}}} \]
\[ = e^{-y^2/2} \cdot \frac{2}{3 n^{1/2}} e^{n^{1/2} \frac{|y|^3/n^{3/2}}{1 - |y|/\sqrt{n}}} \]

Use the remark after Lemma 3
\[ = e^{-y^2/2} \cdot \frac{2}{3 n^{1/2}} e^{n^{1/2} \frac{|y|^3/n^{3/2}}{1 - |y|/\sqrt{n}}} \]

Combine the exponents and coarsely over-estimate the fraction
\[ \leq \frac{|y|^3}{\sqrt{n}} e^{-y^2/6}. \]

\[ \square \]

**Lemma 6.**
\[ \lim_{n \to \infty} g_n(y) = e^{-y^2/2} \]
where the limit is uniform on compact subsets of \( \mathbb{R} \).

**Proof.** For \( |y| \leq \frac{\sqrt{n}}{2} \) use Lemma 5
\[ |g_n(y) - e^{-y^2/2}| \leq \frac{|y|^3}{\sqrt{n}} e^{-y^2/6}. \]

Note that \( |y|^3 e^{-y^2/6} \) has a maximum value of \( 27e^{-3/2} \).

Let \( K \) be a compact subset of \( \mathbb{R} \) with bound \( M \) so that \( K \subset \{ x : |x| < M \} \).
Let \( \epsilon > 0 \) be given. Let \( n \) be so large that \( M < \sqrt{n}/2 \) and \( 27e^{-3/2}/\sqrt{n} < \epsilon \).
Then for all \( y \in K \), \( |g_n(y) - e^{-y^2/2}| \leq \frac{|y|^3}{\sqrt{n}} e^{-y^2/6} \leq 27e^{-3/2}/\sqrt{n} < \epsilon. \)

\[ \square \]

**Lemma 7.** For \( x > -1 \)
\[ \log(1 + x) \leq x - \frac{5}{6} x^2 + \frac{x}{2 + x}. \]
Figure 2: A graph of the functions in Lemma 7 with \( \log(1 + x) \) in red and \( x - \frac{5}{6} \frac{x^2}{(2 + x)} \) in blue.

Remark. Figure 2 is an illustration of the inequality.

Proof. Consider \( f(x) = x - \frac{5}{6} \frac{x^2}{2 + x} - \log(1 + x) \), with \( f'(x) = x \cdot \frac{x^2 - x + 4}{6(1 + x)(2 + x)^2} \).
The only critical point on the domain \( x > -1 \) is \( x = 0 \), and \( f'(x) < 0 \) for \( -1 < x < 0 \), \( f'(x) > 0 \) for \( x > 0 \). Then \( f(x) \) has a global minimum value of 0 at \( x = 0 \). Hence \( \log(x) \leq x - \frac{5}{6} \frac{x^2}{2 + x} \).

Remark. Unfortunately, although the proof is straightforward the origin of the rational function on the right is unmotivated.

Lemma 8.

\[
0 \leq g_n(y) \leq e^{-|y|/6}, \quad |y| > \frac{1}{2} \sqrt{n}
\]

Proof. For \( y > -\sqrt{n} \),

\[
\log(g_n(y)) = n \log(1 + \frac{y}{\sqrt{n}}) - y \sqrt{n}
\leq -\frac{5}{6} \frac{y^2}{2 + y/\sqrt{n}}
\]

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using Lemma 7. For $|y| > \frac{1}{2} \sqrt{n}$ and on the domain $y > -\sqrt{n}$, $1 < 2 + y/\sqrt{n} \leq 2 + |y|/\sqrt{n}$, so

$$\frac{5|y|}{2 + y/\sqrt{n}} \geq \frac{5|y|}{2 + |y|/\sqrt{n}}.$$ 

Now since $|y| > \frac{1}{2} \sqrt{n}$, it is true that $4|y| > 2 \sqrt{n}$, so $5|y| > 2 \sqrt{n} + |y|$ or $\frac{5|y|}{2 + |y|/\sqrt{n}} > \sqrt{n}$. Finally multiplying through by $-|y|/6$,

$$\frac{-5}{6} \frac{y^2}{2 + y/\sqrt{n}} \leq \frac{-5}{6} \frac{y^2}{2 + |y|/\sqrt{n}} < \frac{-|y|\sqrt{n}}{6} < \frac{-|y|}{6}.$$ 

\[ \square \]

**Theorem 9** (Stirling’s Formula).

$$n! \sim \sqrt{2\pi n} n^{n+1/2} e^{-n}.$$ 

*Proof.* By Lemma 2

$$n! = n^n \sqrt{n} e^{-n} \int_{-\infty}^{\infty} g_n(y) \ dy$$ 

where

$$g_n(y) = \left(1 + \frac{y}{\sqrt{n}}\right)^n e^{-y\sqrt{n}} 1_{(-\sqrt{n},\infty)}(y).$$ 

From Lemma 6

$$\lim_{n \to \infty} g_n(y) = e^{-y^2/2}$$ 

uniformly on compact sets and $g_n(y)$ is integrable by Lemma 8. Hence by the Lebesgue Convergence Theorem

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} g_n(y) \ dy = \int_{-\infty}^{\infty} e^{-y^2/2} \ dy$$

and since $\int_{-\infty}^{\infty} e^{-y^2/2} \ dy = \sqrt{2\pi}$ it follows that

$$\lim_{n \to \infty} \frac{n!}{\sqrt{2\pi n^\frac{n}{e^n}}} = \lim_{n \to \infty} \frac{\int_{-\infty}^{\infty} g_n(y) \ dy}{\sqrt{2\pi}} = 1.$$ 

\[ \square \]
Figure 3: A graph of the functions in Lemma 10 with $1 + \frac{1}{12} \cdot \frac{x^2}{1+x} - \left(\frac{1}{x} + \frac{1}{2}\right) \log(1+x)$ in red and $\frac{x^4}{120}$ in blue.

**Asymptotic Expansions**

**Lemma 10.** For $x > 0$

$$0 < 1 + \frac{1}{12} \cdot \frac{x^2}{1+x} - \left(\frac{1}{x} + \frac{1}{2}\right) \log(1+x) \leq \frac{x^4}{120}.$$  

**Remark.** Unfortunately, although the proof is straightforward the origin of the function on the left is unmotivated.

**Remark.** Figure 3 is an illustration of the inequality.

**Proof.** Let $f(x) = 2x + \frac{1}{6} \cdot \frac{x^3}{1+x} - (x+2) \log(1+x)$ for $x > -1$ so $f'(x) = -\frac{1}{6} \cdot \frac{x^3}{(1+x)^2} + \frac{1}{2} \frac{x^2}{2(1+x)} - \log(1+x) - \frac{x+2}{x+1} + 2$. Then $f(0) = 0$, and $f'(0) = 0$. Further $f''(x) = \frac{1}{3} \cdot \frac{x^3}{(1+x)^3}$ and $f''(x) > 0$ for $x > 0$. Therefore, $f(x) > 0$ for $x > 0$ and dividing through by $2x$ yields the left inequality.

For the right side, consider $h(x) = \frac{x^5}{60} - f(x)$. Again $h(0) = h'(0) = 0$. Further $h''(x) = \frac{x^3}{9} - \frac{1}{3} \cdot \frac{x^3}{(1+x)^3} > 0$ for $x > -1$. The right side inequality follows immediately. $$\square$$
Lemma 11. For $x > 0$,

$$e^x \leq 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 e^x.$$ 

Proof. For $x > 0$, expanding $e^x$ in a Taylor series

$$e^x = 1 + x + \frac{1}{2}x^2 + \sum_{k=3}^{\infty} \frac{x^k}{k!}$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \sum_{k=3}^{\infty} \frac{x^{k-3}}{k!/6}$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \sum_{k=0}^{\infty} \frac{x^k}{(k+3)!/6}$$

$$\leq 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

since $(k+3)!/6 \geq k!$ for integers $k \geq 0$. \qed

Lemma 12.

$$\frac{1}{6} e^{1/24} \leq \frac{1}{9940}.$$

Remark. Numerically $\frac{1}{6} e^{1/24} \approx 0.0001005542925$ and $\frac{1}{9940} \approx 0.0001006036217303823$, so the difference is about $5 \times 10^{-8}$.

Proof. I don’t have a rigorous proof based on, say, elementary comparisons in the rational number system. The denominator 9940 is just a convenient “round number” based on numerical evaluation. However, as will be seen in the proof of Theorem 13, what is really needed is that $\frac{1}{6} e^{1/24}$ is a finite constant. Finding a rational number with unit numerator and 4-digit denominator to bound the constant is simply a convenience. \qed

Theorem 13. For $n \geq 2$,

$$\frac{n!}{\sqrt{2\pi}n^{n+1/2}e^{-n}} - 1 - \frac{1}{12n} \leq \frac{1}{288n^2} + \frac{1}{9940n^3}.$$
Proof. Let \( a_n = \frac{n!}{\sqrt{2\pi n^{n+1/2}e^{-n}}} \). Then \( \frac{a_n}{a_{n+1}} = \left(\frac{n+1}{n}\right)^{n+1/2}e^{-1} \) and \( \log\left(\frac{a_n}{a_{n+1}}\right) = (n+1/2)\log(1+1/n) - 1 \). Using the left inequality in Lemma 10 with \( x = 1/n \)

\[
\log\left(\frac{a_n}{a_{n+1}}\right) \leq \frac{1}{12n(n+1)}.
\]

Then

\[
\log\left(\frac{a_n}{a_{n+r}}\right) = \sum_{k=n}^{n+r-1} \log\left(\frac{a_k}{a_{k+1}}\right) \leq \frac{1}{12} \sum_{k=n}^{n+r-1} \frac{1}{k(k+1)} = \frac{1}{12} \left(\frac{1}{n} - \frac{1}{n+r}\right).
\]

Now let \( r \to \infty \), noting that \( \lim_{r \to \infty} a_{n+r} = 1 \), so \( \log(a_n) \leq \frac{1}{12n} \) and \( a_n \leq e^{1/(12n)} \). Now use Lemma 11 and Lemma 12 to yield

\[
a_n \leq 1 + \frac{1}{12n} + \frac{1}{2 (12n)^2} + \frac{1}{6 (12n)^3} e^{1/(12n)}
\]

\[
\leq 1 + \frac{1}{12n} + \frac{1}{2 (12n)^2} + \frac{1}{6 (12n)^3} e^{1/(12n^2)}
\]

\[
\leq 1 + \frac{1}{12n} + \frac{1}{2 (12n)^2} + \frac{1}{9940n^3}.
\]

Then

\[
\frac{n!}{\sqrt{2\pi n^{n+1/2}e^{-n}}} - 1 - \frac{1}{12n} \leq \frac{1}{2 (12n)^2} + \frac{1}{9940n^3}.
\]

Likewise, using the right inequality in Lemma 10 with \( x = 1/n \)

\[
\log\left(\frac{a_n}{a_{n+1}}\right) \geq \frac{1}{12(n(n+1))} - \frac{1}{120n^2}.
\]

Then

\[
\log\left(\frac{a_n}{a_{n+r}}\right) = \sum_{k=n}^{n+r-1} \log\left(\frac{a_k}{a_{k+1}}\right) \geq \frac{1}{12} \sum_{k=n}^{n+r-1} \frac{1}{k(k+1)} - \frac{1}{120} \sum_{k=n}^{n+r-1} \frac{1}{k^4}
\]

\[
= \frac{1}{12} \left(\frac{1}{n} - \frac{1}{n+r}\right) - \frac{1}{120} \sum_{k=n}^{n+r-1} \frac{1}{k^4}.
\]

Again let \( r \to \infty \), noting that \( \lim_{r \to \infty} a_{n+r} = 1 \) to get

\[
\log(a_n) \geq \frac{1}{12n} - \frac{1}{120} \sum_{k=n}^{\infty} \frac{1}{k^4} = \frac{1}{12n} - \frac{1}{120} n - \frac{1}{120} \sum_{k=n+1}^{\infty} \frac{1}{k^4} \geq \frac{1}{12n} - \frac{1}{120} \frac{1}{n^2} - \frac{1}{360n^3}.
\]
because $\sum_{k=n+1}^{\infty} \frac{1}{k^4} \leq \int_{n+1}^{\infty} \frac{1}{x^4} \, dx$ using right-box Riemann sums with width 1.

Let $r_n = \frac{1}{12n} - \frac{1}{120 n^3} - \frac{1}{360 n^4}$. Hence $a_n \geq e^{r_n}$. By a well-known estimate $e^{r_n} \geq 1 + r_n$. Therefore,

$$a_n \geq 1 + \frac{1}{12n} - \frac{1}{120 n^3} - \frac{1}{360 n^4}.$$

Equivalently,

$$\frac{n!}{\sqrt{2\pi n^{n+1/2} e^{-n}}} - 1 - \frac{1}{12n} \geq -\frac{1}{360 n^3} - \frac{1}{120 n^4}.$$

Since

$$\frac{1}{360 n^3} + \frac{1}{120 n^4} \leq \frac{1}{2 (12n)^2} + \frac{1}{9940 n^3},$$

the two inequalities can be summarized as

$$\frac{n!}{\sqrt{2\pi n^{n+1/2} e^{-n}}} - 1 - \frac{1}{12n} \leq \frac{1}{2 (12n)^2} + \frac{1}{9940 n^3}.$$  

\[ \square \]

**Remark.** The proof actually shows the slightly stronger bounds

$$-\frac{1}{360 n^3} - \frac{1}{120 n^4} \leq \frac{n!}{\sqrt{2\pi n^{n+1/2} e^{-n}}} - 1 - \frac{1}{12n} \leq \frac{1}{2 (12n)^2} + \frac{1}{9940 n^3}.$$

**Remark.** Using similar reasoning, from the well-known estimate $e^{r_n} \geq 1 + r_n + \frac{1}{2} r_n^2$, one can derive the estimate

$$\left| \frac{n!}{\sqrt{2\pi n^{n+1/2} e^{-n}}} - 1 - \frac{1}{12n} - \frac{1}{288 n^2} \right| \leq \frac{1}{360 n^3} + \frac{1}{108 n^4}$$

valid for $n \geq 3$.

**Remark.** Note that for $n = 1$,

$$\frac{n!}{\sqrt{2\pi n^{n+1/2} e^{-n}}} = \frac{e}{\sqrt{2\pi}} \geq \frac{13}{12} = 1 + \frac{1}{12}$$

since $\frac{e}{\sqrt{2\pi}} \approx 1.0844$ and $\frac{13}{12} \approx 1.0833$. Similarly for $n = 2$

$$\frac{n!}{\sqrt{2\pi n^{n+1/2} e^{-n}}} = \frac{e^2}{\sqrt{2\pi 2^{3/2}}} \geq \frac{25}{24} = 1 + \frac{1}{12 \cdot 2}$$

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since \( \frac{e^2}{\sqrt{2\pi 2^{3/2}}} \approx 1.042207 \) and \( \frac{25}{3} \approx 1.041666 \). From the remark above, for \( n \geq 3 \)

\[
\frac{n!}{\sqrt{2\pi n^{n+1/2}e^{-n}}} - 1 - \frac{1}{12n} \geq \frac{1}{288n^2} - \frac{1}{360n^3} - \frac{1}{108n^4} > 0.
\]

Then for all \( n \geq 1 \),

\[
\frac{n!}{\sqrt{2\pi n^{n+1/2}e^{-n}}} \geq 1 + \frac{1}{12n},
\]

a better bound than \( e^{1/(2n+1)} \) quoted in equation (1). In order to see this bound, consider

\[ f(x) = \log(1 + x) - \frac{2x}{2 + x}, \]

defined for \( x > -1 \). Then \( f'(x) = \frac{x^2}{(1 + x)(2 + x)^2} \) and \( f(0) = 0 \), \( f''(x) > 0 \) for \( x > 0 \). Then \( \log(1 + \frac{1}{12n}) > \frac{2}{24n+1} > \frac{1}{12n+1} \). Hence \( 1 + \frac{1}{12n} > e^{1/(12n+1)} \).

**Alternate Derivation of Stirling’s Asymptotic Formula**

Note that \( x^t e^{-x} \) has its maximum value at \( x = t \). That is, most of the value of the Gamma function comes from values of \( x \) near \( t \). Therefore use a partition of the Gamma function with \( t > 0 \), \( f(x) = x^t e^{-x} \) for \( x > 0 \), and

\[ A = \{ x : |x - t| \geq t/2 \} \]

Then

\[
\Gamma(t + 1) = \int_0^\infty x^t e^{-x} \, dx
\]

\[ = \int_{t/2}^{3t/2} f(x) \, dx + \int_0^\infty 1_A(x) f(x) \, dx \]

where \( 1_A(\cdot) \) is the indicator function (or characteristic function) of \( A \). For \( x \in A \), \( 1 \leq 4(x - t)^2/t^2 \), so we have \( 1_A(\cdot) \leq 4(x - t)^2/t^2 \). Then

\[
\left| 1 - \frac{1}{\Gamma(t + 1)} \int_{t/2}^{3t/2} x^t e^{-x} \, dx \right| \leq \frac{1}{\Gamma(t + 1)} \int_{\{x : |x - t| \geq t/2\}} \frac{4(x - t)^2}{t^2} x^t e^{-x} \, dx
\]

\[
\leq \frac{4}{\Gamma(t + 1)t^2} \int_0^\infty (x - t)^2 x^t e^{-x} \, dx
\]

Expanding \( (x - t)^2 \) and using \( \Gamma(z + 1) = z\Gamma(z) \) from the remark following Lemma yields

\[
\int_0^\infty (x - t)^2 x^t e^{-x} \, dx = (t + 2)(t + 1)\Gamma(t + 1) - 2t(t + 1)\Gamma(t + 1) + t^2\Gamma(t + 1).
\]
Then simplifying the right side
\[
\left| 1 - \frac{1}{\Gamma(t+1)} \int_{\frac{t}{2}}^{3t/2} x^t e^{-x} \, dx \right| \leq \frac{2 + t}{t^2}
\]

Making the change of variables \( x = y \sqrt{t} + t \) and setting
\[
g_t(y) = (1 + y/\sqrt{t})^t e^{-y \sqrt{t}}
\]
for \( y > -\sqrt{t} \) just as in the proof of Lemma 2.

\[
\lim_{t \to \infty} \frac{t^t \sqrt{t}}{\Gamma(t+1)e^t} \int_{-\sqrt{t}/2}^{\sqrt{t}/2} g_t(y) \, dy = 1. \quad (4)
\]

Now using \( |x| \leq 1/2 \) and Lemma 3
\[
\left| \log(1 + x) - x + \frac{1}{2} x^2 \right| \leq \frac{1}{3} - \frac{|x|^3}{1 - |x|} \leq \frac{2}{3} |x|^3.
\]

Then using Lemma 5
\[
\left| g_t(y) - e^{-y^2/6} \right| \leq \frac{|y|^3}{\sqrt{t}} e^{-y^2/6}
\]
for \( |y| \leq \sqrt{t}/2 \). Then
\[
\left| \int_{-\sqrt{t}/2}^{\sqrt{t}/2} g_t(y) \, dy - \int_{-\infty}^{\infty} e^{-y^2/6} \, dy \right|
\leq \frac{1}{\sqrt{t}} \int_{-\sqrt{t}/2}^{\sqrt{t}/2} |y|^3 e^{-y^2/6} \, dy + \int_{|y| > \sqrt{t}/2} e^{-y^2/2} \, dy
= \frac{-3 \sqrt{t} e^{-t/24}}{2} - \frac{36 e^{-t/24}}{\sqrt{t}} + \frac{36}{\sqrt{t}} + \int_{|y| > \sqrt{t}/2} e^{-y^2/2} \, dy
\]

Therefore,
\[
\lim_{t \to \infty} \int_{-\sqrt{t}/2}^{\sqrt{t}/2} g_t(y) \, dy = \int_{-\infty}^{\infty} e^{-y^2/2} \, dy = \sqrt{2\pi}.
\]
Combining this with the limit in equation 4, we have
\[
\lim_{t \to \infty} \frac{\Gamma(t+1)e^t}{\sqrt{2\pi t^{t+1/2}}} = 1.
\]
Sources
The main part of this section is adapted from the short note by R. Michel, [2]. The alternate derivation is adapted from the even shorter note by R. Michel, [3]. Daniele Malesani corrected some typos and suggested improvements.

Problems to Work for Understanding
1: Show that \( \int_{-\infty}^{\infty} e^{-y^2/2} \, dy = \sqrt{2\pi} \).

Reading Suggestion:

References


Outside Readings and Links:

1. 
2. 
3. 
4. 

Solutions

1: See [Evaluation of the Gaussian Density Integral]

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