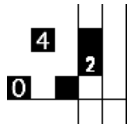


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Stochastic Processes and Advanced Mathematical Finance

Moment Generating Functions



Rating

Mathematically Mature: may contain mathematics beyond calculus with proofs.



Section Starter Question

Give some examples of transform methods in mathematics, science or engineering that you have seen or used and explain why transform methods are useful.



Key Concepts

1. The **moment generating function** converts problems about probabilities and expectations into problems from calculus about function values and derivatives.
 2. The value of the n th derivative of the moment generating function evaluated at 0 is the value of the n th moment of X .
 3. The sum of independent normal random variables is again a normal random variable whose mean is the sum of the means, and whose variance is the sum of the variances.
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Vocabulary

1. The n th **moment** of the random variable X with pdf $f(x)$ is $\mathbb{E}[X^n] = \int_x x^n f(x) dx$ (provided this integral converges absolutely.)
2. The **moment generating function** $\phi_X(t)$ is defined by

$$\phi_X(t) = \mathbb{E}[e^{tX}] = \begin{cases} \sum_i e^{tx_i} p(x_i) & \text{if } X \text{ is discrete} \\ \int_x e^{tx} f(x) dx & \text{if } X \text{ is continuous} \end{cases}$$

for all values t for which the integral converges.



Mathematical Ideas

Transform Methods

We need some tools to aid in proving theorems about random variables. In this section we develop a tool called the **moment generating function** which converts problems about probabilities and expectations into problems from calculus about function values and derivatives. Moment generating functions are one of the large class of transforms in mathematics that turn a difficult problem in one domain into a manageable problem in another domain. Other examples are Laplace transforms, Fourier transforms, Z-transforms, generating functions, and even logarithms.

The general method can be expressed schematically in Figure 1.

Expectation of Independent Random Variables

Lemma 1. *If X and Y are independent random variables, then for any functions g and h :*

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)] \mathbb{E}[h(Y)]$$

Proof. To make the proof definite suppose that X and Y are jointly continuous, with joint probability density function $F(x, y)$. Then:

$$\begin{aligned} \mathbb{E}[g(X)h(Y)] &= \iint_{(x,y)} g(x)h(y)f(x, y) dx dy \\ &= \int_x \int_y g(x)h(y)f_X(x)f_Y(y) dx dy \\ &= \int_x g(x)f_X(x) dx \int_y h(y)f_Y(y) dy \\ &= \mathbb{E}[g(X)] \mathbb{E}[h(Y)]. \end{aligned}$$

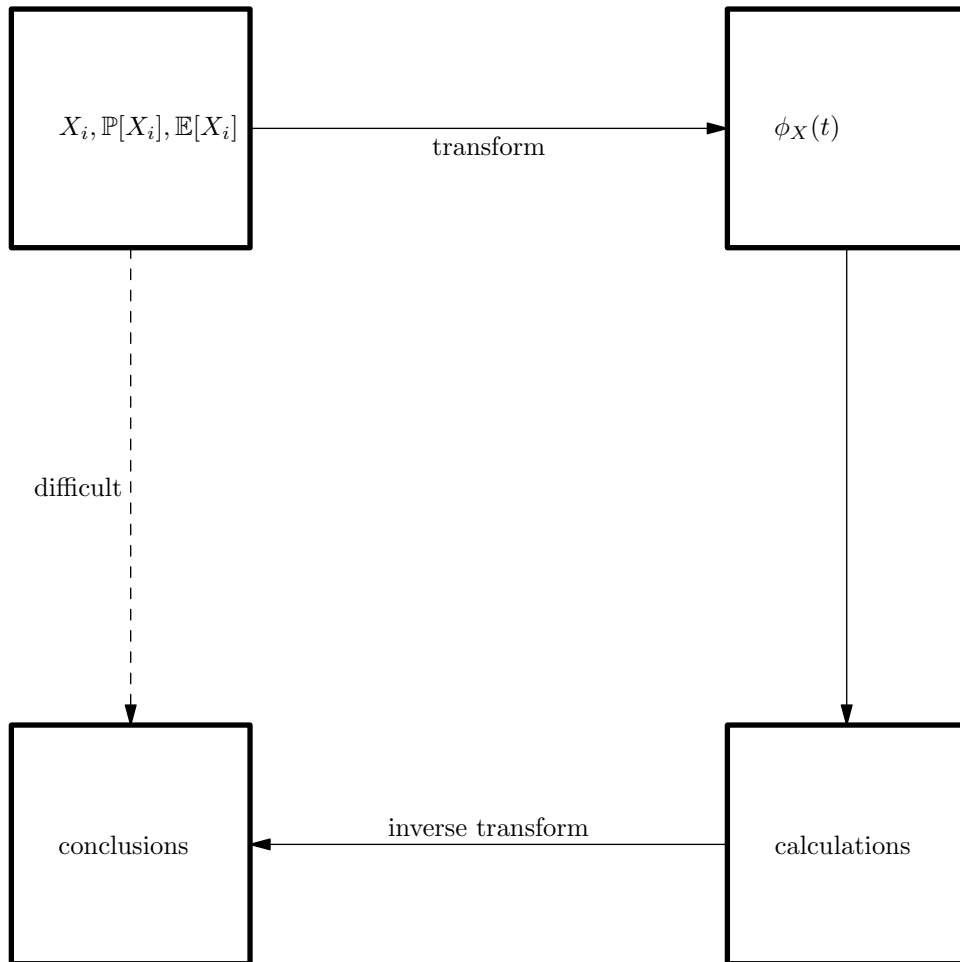


Figure 1: Block diagram of transform methods.

□

Remark. In words, the expectation of the product of *independent* random variables is the product of the expectations.

The Moment Generating Function

The **moment generating function** $\phi_X(t)$ is defined by

$$\phi_X(t) = \mathbb{E} [e^{tX}] = \begin{cases} \sum_i e^{tx_i} p(x_i) & \text{if } X \text{ is discrete} \\ \int_x e^{tx} f(x) dx & \text{if } X \text{ is continuous} \end{cases}$$

for all values t for which the integral converges.

Example. The **degenerate probability distribution** has all the probability concentrated at a single point. That is, the degenerate random variable is a discrete random variable exhibiting certainty of outcome. If X is a degenerate random variable, then $X = \mu$ with probability 1 and X is any other value with probability 0. The moment generating function of the degenerate random variable is particularly simple:

$$\sum_{x_i=\mu} e^{x_i t} = e^{\mu t}.$$

If the moments of order k exist for $0 \leq k \leq k_0$, then the moment generating function is continuously differentiable up to order k_0 at $t = 0$. Assuming that all operations can be interchanged, the moments of X can be generated from $\phi_X(t)$ by repeated differentiation:

$$\begin{aligned} \phi'_X(t) &= \frac{d}{dt} \mathbb{E} [e^{tX}] \\ &= \frac{d}{dt} \int_x e^{tx} f_X(x) dx \\ &= \int_x \frac{d}{dt} e^{tx} f_X(x) dx \\ &= \int_x x e^{tx} f_X(x) dx \\ &= \mathbb{E} [X e^{tX}]. \end{aligned}$$

Then

$$\phi'_X(0) = \mathbb{E}[X].$$

Likewise

$$\begin{aligned}\phi''_X(t) &= \frac{d}{dt}\phi'_X(t) \\ &= \frac{d}{dt} \int_x x e^{tx} f_X(x) dx \\ &= \int_x x \frac{d}{dt} e^{tx} f_X(x) dx \\ &= \int_x x^2 e^{tx} f_X(x) dx \\ &= \mathbb{E}[X^2 e^{tX}].\end{aligned}$$

Then

$$\phi''_X(0) = \mathbb{E}[X^2].$$

Continuing in this way:

$$\phi_X^{(n)}(0) = \mathbb{E}[X^n]$$

Remark. In words, the value of the n th derivative of the moment generating function evaluated at 0 is the value of the n th moment of X .

Theorem 2. *If X and Y are independent random variables with moment generating functions $\phi_X(t)$ and $\phi_Y(t)$ respectively, then $\phi_{X+Y}(t)$, the moment generating function of $X + Y$ is given by $\phi_X(t)\phi_Y(t)$. In words, the moment generating function of a sum of independent random variables is the product of the individual moment generating functions.*

Proof. Using the lemma on independence above:

$$\begin{aligned}\phi_{X+Y}(t) &= \mathbb{E}[e^{t(X+Y)}] \\ &= \mathbb{E}[e^{tX} e^{tY}] \\ &= \mathbb{E}[e^{tX}] \mathbb{E}[e^{tY}] \\ &= \phi_X(t)\phi_Y(t).\end{aligned}$$

□

Theorem 3. *If the moment generating function is defined in a neighborhood of $t = 0$ then the moment generating function uniquely determines the probability distribution. That is, there is a one-to-one correspondence between the moment generating function and the distribution function of a random variable, when the moment-generating function is defined and finite.*

Proof. This proof is too sophisticated for the mathematical level we have now. □

The moment generating function of a normal random variable

Theorem 4. *If $Z \sim N(\mu, \sigma^2)$, then $\phi_Z(t) = \exp(\mu t + \sigma^2 t^2/2)$.*

Proof.

$$\begin{aligned}\phi_Z(t) &= \mathbb{E} [e^{tX}] \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{tx} e^{-(x-\mu)^2/(2\sigma^2)} dx \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(\frac{-(x^2 - 2\mu x + \mu^2 - 2\sigma^2 tx)}{2\sigma^2}\right) dx\end{aligned}$$

By completing the square:

$$\begin{aligned}x^2 - 2\mu x + \mu^2 - 2\sigma^2 tx &= x^2 - 2(\mu + \sigma^2 t)x + \mu^2 \\ &= (x - (\mu + \sigma^2 t))^2 - (\mu + \sigma^2 t)^2 + \mu^2 \\ &= (x - (\mu + \sigma^2 t))^2 - \sigma^4 t^2 - 2\mu\sigma^2 t.\end{aligned}$$

So returning to the calculation of the m.g.f.

$$\begin{aligned}\phi_Z(t) &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(\frac{-((x - (\mu + \sigma^2 t))^2 - \sigma^4 t^2 - 2\mu\sigma^2 t)}{2\sigma^2}\right) dx \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{\sigma^4 t^2 + 2\mu\sigma^2 t}{2\sigma^2}\right) \int_{-\infty}^{\infty} \exp\left(\frac{-(x - (\mu + \sigma^2 t))^2}{2\sigma^2}\right) dx \\ &= \exp\left(\frac{\sigma^4 t^2 + 2\mu\sigma^2 t}{2\sigma^2}\right) \\ &= \exp(\mu t + \sigma^2 t^2/2)\end{aligned}$$

□

Theorem 5. If $Z_1 \sim N(\mu_1, \sigma_1^2)$, and $Z_2 \sim N(\mu_2, \sigma_2^2)$ and Z_1 and Z_2 are independent, then $Z_1 + Z_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$. In words, the sum of independent normal random variables is again a normal random variable whose mean is the sum of the means, and whose variance is the sum of the variances.

Proof. We compute the moment generating function of the sum using our theorem about sums of independent random variables. Then we recognize the result as the moment generating function of the appropriate normal random variable.

$$\begin{aligned}\phi_{Z_1+Z_2}(t) &= \phi_{Z_1}(t)\phi_{Z_2}(t) \\ &= \exp(\mu_1 t + \sigma_1^2 t^2 / 2) \exp(\mu_2 t + \sigma_2^2 t^2 / 2) \\ &= \exp((\mu_1 + \mu_2)t + (\sigma_1^2 + \sigma_2^2)t^2 / 2)\end{aligned}$$

□

An alternative visual proof that the sum of independent normal random variables is again a normal random variable using only calculus is available at [The Sum of Independent Normal Random Variables is Normal](#)

Sources

This section is adapted from: *Introduction to Probability Models*, by Sheldon Ross.



Problems to Work for Understanding

1. Calculate the moment generating function of a random variable X having a uniform distribution on $[0, 1]$. Use this to obtain $\mathbb{E}[X]$ and $\text{Var}[X]$.

2. Calculate the moment generating function of a discrete random variable X having a geometric distribution. Use this to obtain $\mathbb{E}[X]$ and $\text{Var}[X]$.
3. Calculate the moment generating function of a Bernoulli random variable having value 1 with probability p and value 0 with probability $1 - p$. From this, derive the moment generating function of a binomial random variable with parameters n and p .



Reading Suggestion:

References

- [1] Sheldon M. Ross. *Introduction to Probability Models*. Academic Press, 9th edition, 2006.
- [2] S. R. S. Varadhan. *Probability Theory*. Number 7 in Courant Lecture Notes. American Mathematical Society, 2001.



Outside Readings and Links:

1. Generating Functions in Virtual Laboratories in Probability
2. Moment Generating Functions in MathWorld.com

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