1. (a) For $T_0 = 10$ and $a = 20$, draw a graph of the expected duration of a coin-flipping game until victory or ruin as a function of the probability $q$.

(b) For $a = 20$ and $q = 0.55$ draw a graph of the expected duration of a coin-flipping game until victory or ruin as a function of $T_0$.

(c) For $a = 20$ and $q = 0.45$ draw a graph of the expected duration of a coin-flipping game until victory or ruin as a function of $T_0$.

2. This problem is adapted from *Stochastic Calculus and Financial Applications* by J. Michael Steele, Springer, New York, 2001, Chapter 1, Section 1.6, page 9. Information on buy-backs is adapted from investorwords.com. This problem suggests how results on biased random walks can be worked into more realistic models.

Consider a naive model for a stock that has a support level of $21$/share because of a corporate buy-back program. (This means the company will buy back stock if shares dip below $21$ per share. In the case of stocks, this reduces the number of shares outstanding, giving each remaining shareholder a larger percentage ownership of the company. This is usually considered a sign that the company’s management is optimistic about the future and believes that the current share price is undervalued. Reasons for buy-backs include putting unused cash to use,
raising earnings per share, increasing internal control of the company, and obtaining stock for employee stock option plans or pension plans.)

Suppose also that the stock price moves randomly with a downward bias when the price is above $21, and randomly with an upward bias when the price is below $21. To make the problem concrete, we let $S_n$ denote the stock price at time $n$, and we express our stock support hypothesis by the assumptions that

\[
\Pr[S_{n+1} = 22 | S_n = 21] = \frac{9}{10} \\
\Pr[S_{n+1} = 20 | S_n = 21] = \frac{1}{10}
\]

We then reflect the downward bias at price levels above $21$ by requiring that for $k > 21$:

\[
\Pr[S_{n+1} = k + 1 | S_n = k] = \frac{1}{3} \\
\Pr[S_{n+1} = k - 1 | S_n = k] = \frac{2}{3}
\]

We then reflect the upward bias at price levels below $21$ by requiring that for $k < 21$:

\[
\Pr[S_{n+1} = k + 1 | S_n = k] = \frac{3}{4} \\
\Pr[S_{n+1} = k - 1 | S_n = k] = \frac{1}{4}
\]

Using the methods of “single-step analysis” write and solve a set of linear equations that allow you to calculate the expected time for the stock to fall from $25$ through the support level all the way down to $18$.

3. Fix a value $k$. Show that a particular solution $W^p_{sk}$ to the non-homogeneous equation

\[
W^p_{sk} = \delta_{sk} + \frac{1}{2} W^p_{s-1,k} + \frac{1}{2} W^p_{s+1,k}
\]

is

\[
W^p_{sk} = \begin{cases} 
0 & \text{if } s \leq k \\
2k - 2s & \text{if } s > k.
\end{cases}
\]
4. Show that

\[ W_s = \sum_{k=1}^{s-1} kW_{sk} \]

\[ = 2 \left[ \frac{s}{S} \sum_{k=1}^{s-1} k(S - k) - \sum_{k=1}^{s-1} k(s - k) \right] \]

\[ = 2 \left[ \frac{s}{S} \left[ \frac{S(S - 1)(S + 1)}{6} \right] - \frac{s(s - 1)(s + 1)}{6} \right] \]

\[ = \frac{s}{3} [S^2 - s^2] \]

You will need formulas for \( \sum_{k=1}^{N} k \) and \( \sum_{k=1}^{N} k^2 \) or alternatively for \( \sum_{k=1}^{N} k(M - k) \). These are easily found or derived.

5. (a) For the long run average cost

\[ C = \frac{K + (1/3)rS^3x(1 - x^2)}{S^2x(1 - x)}. \]

find \( \partial C/\partial x \).

(b) For the long run average cost

\[ C = \frac{K + (1/3)rS^3x(1 - x^2)}{S^2x(1 - x)}. \]

find \( \partial C/\partial S \).

(c) Find the optimum values of \( x \) and \( S \).