1. The current exchange rate between the U.S. Dollar (USD) and the Great Britain Pound (GBP) is 1.5641, that is, it costs $1.5641 to buy one Pound. Take the Fed Funds rate, (technically the bank-to-bank overnight lending rate), in the United States to be approximately 0.20% (assume it is compounded continuously). The corresponding bank-to-bank lending rate in Great Britain is the LIBOR and it currently is approximately 0.26% (assume it is compounded continuously). What is the forward rate (the exchange rate in a forward contract that allows you to buy Pounds in a year) for purchasing Pounds 1 year from today. What principle allows you to claim that value?

Solution: Purchase a forward contract to change GBP to USD in one year’s time at rate $a$ to be determined. Assume that you borrow $X$ USD and then exchange it for $X/1.5641$ GBP. You then invest it in banks in Great Britain for a year obtaining $X \exp(0.0026 \cdot 1)/1.5641$. Then you can exercise your forward contract and change the GBP to USD, obtaining $aX \exp(0.0026 \cdot 1)/1.5641$. Then you have to pay back the loan in the amount $X \exp(0.002 \cdot 1)$. These amounts should be the same or an arbitrage opportunity exists. Then

$$a \frac{\exp(0.0026)}{1.5641} = \exp(0.0020)$$

or $a = 1.5632$. 

1
2. According to the article “Bullion bulls” on page 81 in the October 8, 2009 issue of *The Economist*, gold has risen from about $510 per ounce in January 2006 to about $1050 per ounce in October 2009, 46 months later. In September 2010 it is about $1300.

(a) What is the continuously compounded annual rate of increase of the price of gold over the period January 2006 to October 2009?

(b) What is the continuously compounded annual rate of increase of the price of gold over the period October 2009 to September 2010?

(c) In October 2009, one can borrow or lend money at 5% interest, again assume it compounded continuously. In view of this, describe a strategy that will make a profit in October 2010, involving borrowing or lending money, assuming that the rate of increase in the price gold stays constant over this time.

(d) The article suggests that the rate of increase for gold will stay constant. Did it? In view of this, what do you expect to happen to interest rates and what principle allows you to conclude that? Did they?

**Solution:**

(a) The continuously compounded annual rate of increase of gold over this period is the solution $r$ of the equation

$$g(T) = g(0) \exp(rT)$$

where $T = 46/12$, $g(T) = 1050$, $g(0) = 510$, or

$$r = \frac{\ln(g(T)/g(0))}{T} = \frac{\ln(1050/510)}{(46/12)} = 0.1883829697,$$

about 18.8% annual rate.

(b) Same calculation with $T = 11/12$, $g(0) = 1300$, and $g(0) = 1050$. Then

$$r = \frac{\ln(g(T)/g(0))}{T} = \frac{\ln(1300/1050)}{(11/12)} = 0.23299,$$

about 23.3% annual rate.
(c) One could borrow $1050 at 5% continuously compounded interest and buy one ounce of gold at $1050. Holding the gold for one year, its value in October 2010 is $1050 \times e^{0.05} = 1267.66$. You could sell the gold, making $217.66$ per ounce. You then repay the loan which will require $1050 \times e^{0.05} = 1103.83$, or approximately $53.83$ in interest. The profit per ounce of gold is $217.66 - 53.83 = 163.83$.

(d) The principle of arbitrage says that this opportunity will not stay for long, so if the rate of increase of gold stays constant (or even increases as it did over the last year!), then interest rates must rise. However, interest rates actually fell over the period. The conclusion is not that the principle of no-arbitrage opportunities is incorrect, but rather that the true market is more complex than a simple market consisting only of gold and bonds (loans). The true market has other factors keeping rates low.

3. Suppose that there is a 20% decrease in the default rate from 5% to 4%. By what factor do the default rates of the 10-tranches and the derived 10th CDO change?

**Solution:** With a decrease in base rate of default of 20% from $p = 0.05$ to $p = 0.04$, the default rate for the 10-tranche decreases from $0.028188$ to $0.0068445$, a decrease by a factor of $0.24281$.

```
octave:15> 1 - binocdf(9,100, 0.05) ans = 0.028188
octave:16> t10p5 = 1 - binocdf(9,100, 0.05) t10p5 = 0.028188
octave:17> t10p4 = 1 - binocdf(9,100, 0.04) t10p4 = 0.0068445
octave:18> t10p4/t10p5 ans = 0.24281
```

With a decrease in base rate of default of 20% from $p = 0.05$ to $p = 0.04$, the default rate for the 10-CDO decreases from $5.4389 \times 10^{-4}$ to $2.2300 \times 10^{-9}$, a decrease by a factor of $4.1001 \times 10^{-6}$.

```
octave:20> cdo10p5 = 1 - binocdf(9,100, t10p5)
```
4. For the tranches create a table of probabilities of default for tranches $i = 5$ to $i = 15$ for probabilities of default $p = 0.03, 0.04, 0.05, 0.06$ and 0.07 and determine where the tranches become safer investments than the individual mortgages on which they are based.

**Solution:** Here is one way to get a table (or matrix) of default probabilities for the tranches:

```octave
octave:14> [[ 1- binocdf([4:14], 100, 0.03)];
    [1- binocdf([4:14], 100,0.04)];
    [ 1- binocdf([4:14], 100, 0.05)];
    [1- binocdf([4:14], 100,0.06)];
    [ 1- binocdf([4:14], 100, 0.07))]
```

ans =

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5. For a base mortgage default rate of 5%, draw the graph of the default rate of the tranches as a function of the tranche number.

**Solution:** To graph the default rate as a function of the tranche number, the following command in Octave will suffice: `plot( 1 - binocdf([0:100], 100, 0.05))` and the resulting plot is:

To get a better look at the interesting part of the graph from tranche 0 to tranche 11: `plot( 1 - binocdf([0:11], 100, 0.05))` and the resulting close-up is:
Figure 1: Default rate as a function of the tranche number, for $n = 100$ and $p = 0.05$. 
Figure 2: Close-up of the default rate as a function of the tranche number, for $n = 100$ and $p = 0.05$. 