1. A second-order, linear ordinary differential equation is one of the form

$$\alpha(t)u'' + \beta(t)u' + \gamma(t)u = f(t). \tag{1}$$

We assume that $\alpha(t) \neq 0$ for t in an open interval I. We can thus write the equation as

$$u'' + p(t)u' + q(t)u = g(t), \tag{2}$$

where

$$p(t) = \frac{\beta(t)}{\alpha(t)}, \quad q(t) = \frac{\gamma(t)}{\alpha(t)}, \quad \text{and} \quad g(t) = \frac{f(t)}{\alpha(t)}.$$

If $g(t) \equiv 0$, the equation is called *homogeneous.*

2. Let $D = d/dt$, $D^2 = d^2/dt^2$ and $L = D^2 + p(t)D + q(t)$. Thus

$$\begin{aligned} Lu &= \big(D^2 + p(t)D + q(t)\big)\, u \\ &= D^2u + p(t)Du + q(t)u \\ &= u'' + p(t)u' + q(t)u. \end{aligned}$$

L is called a second-order, linear *differential operator.* Using operator notation, we can write the linear inhomogeneous equation as

$$Lu = g(t), \tag{3}$$

and the homogeneous equation as

$$Lu = 0. \tag{4}$$

3. **Proposition:** The function $z(t) = 0$ satisfies the homogeneous equation (4).

4. **Proposition:** For any constant c, and functions $\varphi(t)$ and $\psi(t)$,

$$L[c\varphi] = cL\varphi, \tag{5}$$

and

$$L[\varphi + \psi] = L\varphi + L\psi. \tag{6}$$

5. **Proposition:** If $u_1(t)$ and $u_2(t)$ satisfy (4) for all t in an open interval I, then for any constants c_1 and c_2, the function

$$u(t) = c_1u_1(t) + c_2u_2(t)$$

does too. This is called the principle of *superposition.*

6. **Example:** Let $L = D^2 + 4$. Thus the homogeneous equation (4) is

$$Lu = u'' + 4u = 0. \tag{7}$$

The functions $u_1 = \cos 2t$ and $u_2 = \sin 2t$ are solutions to (7). By the superposition principle, any function of the form

$$u(t) = c_1 \cos 2t + c_2 \sin 2t$$

is also.

7. **Proposition:** If $p(t)$, $q(t)$ and $g(t)$ are continuous on the open interval I, then for any $\tau \in I$, the initial value problem

$$\begin{cases} Lu = g(t), & \text{for } t \in I, \\ u(\tau) = a, \\ u'(\tau) = b, \end{cases} \tag{8}$$

has a unique solution.

8. **Note:** From now on, we will assume the continuity of $p(t)$, $q(t)$ and $g(t)$ on some open interval (possibly the whole real line) I. It will be understood that solutions to the linear equation satisfy the equation on I.

9. **Definition:** Solutions $u_1(t)$ and $u_2(t)$ to the homogeneous equation (4) are *linearly independent* or simply *independent* if neither is a constant multiple of the other. If one is a constant multiple of the other, then they are *linearly dependent* or simply *dependent.*

10. Consider the homogeneous equation $u'' - u = 0$. (Here, $L = D^2 - 1$.) The functions

$$u(t) = e^t, \quad v(t) = -e^t, \quad w(t) = e^{-t} \quad \text{and} \quad z(t) = 0,$$

are all solutions. It isn't hard to see that

a. u and v are dependent,

b. u and w are independent,

c. v and w are independent,

d. z and any one of u, v and w are dependent.

11. **Note:** By proposition (3), the homogeneous equation (4) always has $z(t) = 0$ as a solution. Clearly, $z(t)$ and any other solution are linearly dependent.

12. **Definition:** The *Wronskian determinant* of $u_1(t)$ and $u_2(t)$ is

$$\begin{aligned} W(u_1, u_2)(t) &= \begin{vmatrix} u_1(t) & u_2(t) \\ u_1'(t) & u_2'(t) \end{vmatrix} \\ &= u_1(t)u_2'(t) - u_2(t)u_1'(t). \end{aligned} \tag{9}$$

13. **Proposition:** If $u_1(t)$ and $u_2(t)$ are linearly dependent, then $W(u_1, u_2)(t) \equiv 0$.

14. The above proposition gives us an easy way to test for linear independence: If $W(u_1, u_2)(\tau) \neq 0$ at even a single point τ, then $u_1(t)$ and $u_2(t)$ are independent.

15. **Proposition:** If $u_1(t)$ and $u_2(t)$ are linearly independent solutions to the homogeneous equation (4), then $W(u_1, u_2)(t) \neq 0$ for all t.

16. **Proposition:** If $u_1(t)$ and $u_2(t)$ are linearly independent solutions to the homogeneous equation (4), then the general solution to (4) is

$$u(t) = c_1 u_1(t) + c_2 u_2(t). \tag{10}$$

In other words, every solution to (4) is of the form (10) for some constants c_1 and c_2.

17. **Example:** Consider the homogeneous equation (7). Two solutions are $u_1(t) = \cos 2t$ and $u_2(t) = \sin 2t$. That they are independent can be checked either by inspection or by using the Wronskian: Since $W(u_1, u_2)(t) = 2$, they are indeed independent. Then by proposition (16), the general solution to (7) is

$$u(t) = c_1 \cos 2t + c_2 \sin 2t. \tag{11}$$

18. Propostion (16) provides us a recipe for solving the initial value problem for the homogeneous equation,

$$\begin{cases} Lu = 0, & \text{for } t \in I, \\ u(\tau) = a, \\ u'(\tau) = b. \end{cases} \tag{12}$$

a. Find two linearly independent solutions u_1 and u_2 to $Lu = 0$ on I. You can check linear independence either by inspection or with the Wronskian.

b. Form the general solution

$$u(t) = c_1 u_1(t) + c_2 u_2(t), \tag{13}$$

and then use the initial conditions to determine c_1 and c_2.

19. **Example:** Solve the initial value problem

$$\begin{cases} u'' - 3u' + 2u = 0, \quad \text{for } t \in \mathbf{R}, \\ u(0) = 2, \\ u'(0) = -4. \end{cases} \tag{14}$$

The functions $u_1(t) = e^t$ and $u_2(t) = e^{2t}$ are linearly independent solutions to the ODE. Hence the general solution is

$$u(t) = c_1 e^t + c_2 e^{2t}. \tag{15}$$

And thus

$$\begin{cases} u(0) = 2 = c_1 + c_2, \\ u'(0) = -4 = c_1 + 2c_2. \end{cases}$$

The solution to this system of equations is $(c_1, c_2) = (8, -6)$, and the solution to the initial value problem (14) is

$$u(t) = 8e^t - 6e^{2t}.$$

20. **Example:** Solve the initial value problem

$$\begin{cases} u'' + \frac{3}{t}u' - \frac{3}{t^2}u = 0, \quad \text{for } t > 0 \\ u(1) = 0, \\ u'(1) = 2. \end{cases} \tag{16}$$

The functions $u_1(t) = t$ and $u_2(t) = t^{-3}$ are linearly independent solutions to the ODE for $t > 0$. Hence the general solution is

$$u(t) = c_1 t + c_2 t^{-3}. \tag{17}$$

And thus

$$\begin{cases} u(1) = 0 = c_1 + c_2, \\ u'(1) = 2 = c_1 - 3c_2. \end{cases}$$

The solution to this system of equations is $(c_1, c_2) = (.5, -.5)$, and the solution to the initial value problem (16) is

$$u(t) = .5t - .5t^{-3}.$$

21. **Definition:** A pair $\{u_1, u_2\}$ of linearly independent solutions to the homogeneous equation (4) is called a *fundamental set* for the equation. So, for example, $\{e^t, e^{2t}\}$ is a fundamental set for the equation $u'' - 3u' + 2u = 0$.