The Laplace Transform 1

1. The *Laplace transform* of a function $f(t)$ is

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt, \tag{1}$$

defined for those values of s at which the integral converges. For example, the Laplace transform of $f(t) = e^{at}$ is

$$\begin{aligned} \mathcal{L}\left\{e^{at}\right\} &= \int_0^\infty e^{-st} e^{at}\, dt \\ &= \int_0^\infty e^{-(s-a)t}\, dt \\ &= (s-a)^{-1}, \quad \text{for } s > a. \end{aligned} \tag{2}$$

2. Note that the Laplace transform of $f(t)$ is a function of s. Hence the transform is sometimes denoted $\mathcal{L}\{f(t)\}(s)$, $\mathcal{L}\{f\}(s)$, or simply $F(s)$.

3. **Example**: The Laplace transform of $f(t) = 1$ is

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st}\, dt \\ &= s^{-1}, \quad \text{for } s > 0. \end{aligned} \tag{3}$$

You can integrate by parts obtain the Laplace transform of $f(t) = t$:

$$\begin{aligned} \mathcal{L}\{t\} &= \int_0^\infty t e^{-st}\, dt \\ &= s^{-2}, \quad \text{for } s > 0. \end{aligned} \tag{4}$$

Integrate by parts n times to get

$$\begin{aligned} \mathcal{L}\{t^n\} &= \int_0^\infty t^n e^{-st}\, dt \\ &= \frac{n!}{s^{n+1}}, \quad \text{for } s > 0, \text{ and } n = 0, 1, 2, \ldots \end{aligned} \tag{5}$$

4. The Gamma function is

$$\Gamma(x) = \int_0^\infty z^{x-1} e^{-z}\, dz, \quad \text{for } x > 0. \tag{6}$$

We showed in class that

$$\Gamma(x+1) = x\Gamma(x). \tag{7}$$

Thus $\Gamma(x)$ is the continuous extension of the factorial. We also showed that for $a > -1$,

$$\mathcal{L}\{\Gamma(t)\} = \frac{\Gamma(a+1)}{s^{a+1}}, \quad \text{for } s > 0. \tag{8}$$

As $\Gamma(x)$ generalizes the factorial, the Laplace transform (8) generalizes (5).

5. **Example**: The Laplace transforms of $\sin\beta t$ and $\cos\beta t$ are

$$\begin{aligned}\mathcal{L}\{\sin\beta t\} &= \int_0^\infty e^{-st}\sin\beta t\,dt \\ &= \frac{\beta}{s^2+\beta^2},\end{aligned} \tag{9}$$

and

$$\begin{aligned}\mathcal{L}\{\cos\beta t\} &= \int_0^\infty e^{-st}\cos\beta t\,dt \\ &= \frac{s}{s^2+\beta^2},\end{aligned} \tag{10}$$

both for $s > 0$.

6. **Proposition**: The Laplace transform is a linear operator, that is, for functions $f(t)$ and $g(t)$ and any constant c,

$$\mathcal{L}\{f(t)+g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}, \tag{11}$$

and

$$\mathcal{L}\{cf(t)\} = c\mathcal{L}\{f(t)\}. \tag{12}$$

7. **Example**: From (11), (12) and (2) we get

$$\mathcal{L}\{\sinh\beta t\} = \frac{\beta}{s^2-\beta^2}, \tag{13}$$

and

$$\mathcal{L}\{\cosh\beta t\} = \frac{s}{s^2-\beta^2}, \tag{14}$$

both for $s > |\beta|$.

8. Example: The Laplace transform of $x(t) = 2t - e^{-3t} + 4\cos\pi t$ is

$$\begin{aligned} X(s) &= \mathcal{L}\{2t - e^{-3t} + 4\cos\pi t\} \\ &= 2\mathcal{L}\{t\} - \mathcal{L}\{e^{-3t}\} + 4\mathcal{L}\{\cos\pi t\} \\ &= \frac{2}{s^2} - \frac{1}{s+3} + \frac{4s}{s^2+\pi^2}. \end{aligned} \tag{15}$$

9. When applying the Laplace transform to a function $f(t)$, we will assume that f is of *exponential order* over $[0, \infty)$. This means that for some t_0, and constants M and α,

$$|f(t)| \leq Me^{\alpha t}, \tag{16}$$

for all $t \geq t_0$. In class we proved the following result.

10. Proposition: If $f(t)$ satisfies (16) then its Laplace transform $F(s)$ exists for $s > \alpha$ and

$$\lim_{s\to\infty} F(s) = 0. \tag{17}$$

11. Note: Unless otherwise stated, we'll assume any function to which we apply the Laplace transform to be of exponential order. We conclude this set of notes with a few important properties of the Laplace transform. The derivations, done in class, are quite simple.

12. The shift property: Let $\mathcal{L}\{f(t)\} = F(s)$. Then

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a). \tag{18}$$

So, for example,

$$\mathcal{L}\{e^{-t}\cos 2t\} = \frac{s+1}{(s+1)^2+4}.$$

13. The switching property: Let $H(t)$ be the Heaviside function:

$$H(t) = \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t \geq 0, \end{cases}$$

and $F(s)$ be the Laplace transform of $f(t)$. Then

$$\mathcal{L}\{H(t-a)f(t-a)\} = e^{-as}F(s). \tag{19}$$

Thus,

$$\mathcal{L}\{H(t-3)(t-3)^5\} = \frac{5!\,e^{-3s}}{s^6}.$$

And,

$$\begin{aligned}\mathcal{L}\{H(t-3)t^2\} &= \mathcal{L}\{H(t-3)(t-3+3)^2\}\\ &= \mathcal{L}\left\{H(t-3)[(t-3)^2+6(t-3)+9]\right\}\\ &= e^{-3s}\left[\frac{2}{s^3}+\frac{6}{s^2}+\frac{9}{s}\right]. \end{aligned} \tag{20}$$

Note: Some authors write $h_a(t)$ or $u_a(t)$ instead of $H(t-a)$.

14. If $F(s)$ is the Laplace transform of $f(t)$, and n is a nonnegative integer, then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s). \tag{21}$$

Thus,

$$\mathcal{L}\{t \sin 2t\} = \frac{4s}{(s^2+4)^2}.$$

15. Let $F(s) = \mathcal{L}\{f(t)\}$. Then

$$\mathcal{L}\{f'(t)\} = -f(0) + sF(s), \tag{22}$$

and

$$\mathcal{L}\{f''(t)\} = -sf(0) - f'(0) + s^2F(s). \tag{23}$$

16. The *convolution* of functions $f(t)$ and $g(t)$ is

$$(f*g)(t) = \int_0^t f(t-u)g(u)\,du. \tag{24}$$

As we showed in class, convolution is commutative, i.e.

$$(f*g)(t) = \int_0^t f(t-u)g(u)\,du = \int_0^t f(u)g(t-u)\,du = (g*f)(t). \tag{25}$$

17. **Proposition**: If $F(s)$ and $G(s)$ are the Laplace transforms of $f(t)$ and $g(t)$ respectively, then

$$\mathcal{L}\{(f*g)(t)\} = F(s)G(s). \tag{26}$$

18. We can use the above proposition to compute the Laplace transform of $\int_0^t f(u)\,du$. We can write this integral as $(f*1)(t)$ and then apply (26):

$$\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \mathcal{L}\{(f*1)(t)\} = \frac{F(s)}{s}. \tag{27}$$