Improper Integrals

1. An integral is *improper* if the integrand, the domain of integration, or both, are unbounded.

2. If $f(x)$ is continuous on $[a,b)$ and $f(x) \to \pm\infty$ as $x \to b^-$, then by definition,

$$\int_a^b f(x)\,dx = \lim_{r \to b^-} \int_a^r f(x)\,dx. \tag{1}$$

3. If $f(x)$ is continuous on $[a,b)$ and $f(x) \to \pm\infty$ as $x \to b^-$, then by definition,

$$\int_a^b f(x)\,dx = \lim_{r \to b^-} \int_a^r f(x)\,dx. \tag{2}$$

4. If $f(x)$ is continuous on $[a,c) \cup (c,b]$, and unbounded at $x = c$, then

$$\int_a^b f(x)\,dx = \lim_{r \to c^-} \int_a^r f(x)\,dx + \lim_{s \to c^+} \int_s^b f(x)\,dx. \tag{3}$$

The integral on the left-hand side of (3) is convergent if and only if the two on the right-hand side are.

5. If $f(x)$ is continuous on $[a,\infty)$, then by definition,

$$\int_a^\infty f(x)\,dx = \lim_{r \to \infty} \int_a^r f(x)\,dx. \tag{4}$$

6. If $f(x)$ is continuous on $(-\infty,b]$, then by definition,

$$\int_{-\infty}^b f(x)\,dx = \lim_{r \to -\infty} \int_r^b f(x)\,dx. \tag{5}$$

7. If $f(x)$ is continuous on $(-\infty,\infty)$, then by definition,

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{r \to -\infty} \int_r^c f(x)\,dx + \lim_{s \to \infty} \int_c^s f(x)\,dx, \tag{6}$$

where c is any fixed point. The integral on the left-hand side of (6) is convergent if and only if the two on the right-hand side are.

8. For example, if $f(x)$ is continuous on (a,b) and $f(x) \to \pm\infty$ as $x \to a^+$ and $x \to b^-$, then we define

$$\int_a^b f(x)\,dx = \lim_{r \to a^+} \int_r^c f(x)\,dx \lim_{s \to b^-} \int_c^s f(x)\,dx, \tag{7}$$

where c is any point in (a,b). The integral on the left-hand side of (7) is convergent if and only if the two on the right-hand side are.