WKB (Wentzel-Kramers-Brillouin) Approximation

1. We apply the WKB method to approximate solutions to equations of the form

$$\varepsilon^2 y'' + q(x)y = 0, \quad \varepsilon \ll 1, \tag{1}$$

$$y'' + q(\varepsilon x)^2 y = 0, \quad \varepsilon \ll 1, \tag{2}$$

and

$$-y'' + q(x)y = \lambda^2 p(x)y, \quad \lambda \gg 1. \tag{3}$$

2. **Example**: The wave function $\Psi(x,t)$ in one space dimension satisfies the Schrödinger equation

$$i\hbar\Psi_t = -\frac{\hbar^2}{2m}\Psi_{xx} + V(x)\Psi,$$

where V is the potential, m the mass and $\hbar = h/2\pi$, h being Planck's constant. We separate variables by setting

$$\Psi(x,t) = \phi(t)y(x).$$

We set the x and t terms equal to the constant E and obtain the equation

$$\frac{\hbar^2}{2m}y'' + (E - V(x))y = 0, \tag{4}$$

which is the time-independent Schrödinger equation. Set

$$q(x) = E - V(x),$$

and

$$\frac{\hbar^2}{2m} = \varepsilon^2$$

to obtain an equation of the form (1).

3. **The Nonoscillatory Case**: If $q(x) < 0$ over the interval of interest, set $q(x) = -k(x)^2$, where $k(x) > 0$. The equation becomes

$$\varepsilon^2 y'' - k(x)^2 y = 0. \tag{5}$$

Were $k(x) \equiv k_0$, a real constant, then (5) would have linearly independent, nonoscillatory solutions of the form $\exp(\pm k_0 x/\varepsilon)$. This suggests the change of variable

$$y = e^{\frac{u(x)}{\varepsilon}}. \tag{6}$$

The function u satisfies the equation

$$\varepsilon u'' + {u'}^2 - k(x)^2 = 0.$$

We set $u' = v$ to get

$$\varepsilon v' + v^2 - k(x)^2 = 0. \tag{7}$$

4. Plug the regular perturbation expansion

$$v = v_0 + \varepsilon v_1 + \cdots.$$

into (7):

$$\varepsilon(v_0 + \varepsilon v_1 + \cdots)' + (v_0 + \varepsilon v_1 + \cdots)^2 - k(x)^2 = 0.$$

Matching powers of ε gives the equations

$$\begin{aligned} O(1): &\quad v_0^2 - k(x)^2 = 0, \\ O(\varepsilon): &\quad 2v_0 v_1 = -v_0'. \end{aligned}$$

From the $O(1)$ equation we obtain

$$v_0(x) = \pm k(x).$$

Put this in the $O(\varepsilon)$ equation to get

$$v_1(x) = -\frac{k'(x)}{2k(x)}.$$

Thus v has the expansion

$$v(x) = \pm k(x) - \varepsilon \frac{k'(x)}{2k(x)} + O(\varepsilon^2). \tag{8}$$

And since $v = u'$,

$$u(x) = \pm \int_\xi^x k(z)\, dz - \frac{\varepsilon}{2} \ln \frac{k(x)}{k(\xi)} + O(\varepsilon^2), \tag{9}$$

where ξ is arbitrary. Thus

$$\begin{aligned} y_\pm(x) &= e^{\frac{u(x)}{\varepsilon}} \\ &= \left[\frac{k(\xi)}{k(x)}\right]^{\frac{1}{2}} e^{\pm\frac{1}{\varepsilon}\int_\xi^x k(z)\, dz}\, e^{O(\varepsilon)} \\ &= \left[\frac{k(\xi)}{k(x)}\right]^{\frac{1}{2}} e^{\pm\frac{1}{\varepsilon}\int_\xi^x k(z)\, dz}\, (1 + O(\varepsilon)). \end{aligned} \tag{10}$$

In (10), we have approximations of two linearly independent solutions to (5), one with the plus sign and the other the minus. Thus, to leading order, any solution y to (5) will have the WKB approximation

$$y(x) \approx y_a(x) = \frac{c_1}{\sqrt{k(x)}} e^{\frac{1}{\varepsilon}\int_\xi^x k(z)\, dz} + \frac{c_2}{\sqrt{k(x)}} e^{-\frac{1}{\varepsilon}\int_\xi^x k(z)\, dz}, \tag{11}$$

for constants c_1 and c_2.

5. **Example**: Find the WKB approximation to the solution of the equation

$$\begin{cases} \varepsilon^2 y'' - (1+x)^2 y = 0, & \text{for } x > 0,\ \varepsilon \ll 1, \\ y(0) = 1, \\ y(\infty) = 0. \end{cases}$$

Take $\xi = 0$. The WKB approximation has the form

$$y_a(x) = \frac{c_1}{\sqrt{1+x}} e^{\frac{1}{\varepsilon}\int_0^x (1+z)\,dz} + \frac{c_2}{\sqrt{1+x}} e^{-\frac{1}{\varepsilon}\int_0^x (1+z)\,dz}. \tag{12}$$

The second boundary condition forces us to take $c_1 = 0$. The first boudnary condition then implies that $c_2 = 1$. Hence, to leading order,

$$\begin{aligned} y(x) &\approx y_a \\ &= \frac{1}{\sqrt{1+x}} e^{-\frac{1}{\varepsilon}\int_0^x (1+z)\,dz} \\ &= \frac{1}{\sqrt{1+x}} e^{-\frac{1}{\varepsilon}\left(x+\frac{x^2}{2}\right)}. \end{aligned} \tag{13}$$

6. **The Oscillatory Case**: When $q(x) > 0$ over the interval of interest, we set $q(x) = k(x)^2$, where $k(x) > 0$. The equation becomes

$$\varepsilon^2 y'' + k(x)^2 y = 0. \tag{14}$$

Were $k(x) \equiv k_0$, a real constant, then (14) would have oscillatory solutions of the form $\exp(\pm i k_0 x/\varepsilon)$. This suggests the change of variable

$$y = e^{\frac{iu(x)}{\varepsilon}}, \tag{15}$$

for some real-valued function $u(x)$. The same analysis as the foregoing yields WKB approximations to linearly independent solutions to equation (14):

$$y_\pm(x) = \frac{1}{\sqrt{k(x)}} e^{\pm\frac{i}{\varepsilon}\int_\xi^x k(z)\,dz}. \tag{16}$$

To finish the derivation, use the Coates-Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$, to rewrite $y_\pm$ terms of sines and cosines. Then use these to form the the WKB approximation to the general solution to (14):

$$y(x) \approx y_a(x) = \frac{c_1}{\sqrt{k(x)}} \sin\left(\frac{1}{\varepsilon}\int_\xi^x k(z)\,dz\right) + \frac{c_2}{\sqrt{k(x)}} \cos\left(\frac{1}{\varepsilon}\int_\xi^x k(z)\,dz\right), \tag{17}$$

for constants c_1 and c_2. We can multiply and divide the above expression by

$$A = (c_1^2 + c_2^2)^{\frac{1}{2}},$$

and then rewrite it as

$$\frac{A}{\sqrt{k(x)}} \cos\left(\frac{1}{\varepsilon}\int_{\xi}^{x} k(z)\,dz - \phi\right), \tag{18}$$

where

$$\phi = \arctan\frac{c_1}{c_2}$$

is the phase.

7. **Example**: Consider the time-independent Schrödinger equation (4). If $E > V(x)$, we have the oscillatory case. Since $\hbar$ is small, we can apply the WKB method obtain the approximate solution

$$y_a(x) = \frac{A}{(E - V(x))^{\frac{1}{4}}} \cos\left(\frac{\sqrt{2m}}{\hbar}\int_{\xi}^{x} \sqrt{E - V(z)}\,dz - \phi\right).$$