Singular Perturbation

1. When the character of the problem changes discontinuously at $\varepsilon = 0$ we have a singular perturbation.

2. Consider the boundary value problem

$$(P)\begin{cases} \varepsilon y'' + (1+\varepsilon)y' + y = 0, & \text{for } 0 < t < 1,\ \varepsilon \ll 1, \\ y(0) = 0, \\ y(1) = 0. \end{cases}$$

The regular perturbation expansion

$$y = y_0 + \varepsilon y_1 + \cdots, \tag{1}$$

leads to the problem

$$O(1)\begin{cases} y_0' + y_0 = 0, & \text{for } 0 < t < 1, \\ y_0(0) = 0, \\ y_0(1) = 0. \end{cases}$$

Since the equation for y_0 is of the first-order, we can't satisfy both boundary conditions. Regular perturbation fails. It is clear that the order of the equation in (P) drops from two to one at $\varepsilon = 0$. Thus the perturbation is singular.

3. The solution to (P) is

$$y(t) = \frac{e^{-t} - e^{-\frac{t}{\varepsilon}}}{e^{-1} - e^{-\frac{1}{\varepsilon}}}. \tag{2}$$

Thus

$$y'(0) = O(\varepsilon^{-1}) \quad \text{and} \quad y''(0) = O(\varepsilon^{-2}), \tag{3}$$

as $\varepsilon \downarrow 0$. Away from $t = 0$ (say, at $t = .5$)

$$y(.5) = O(1), \quad y'(.5) = O(1), \quad \text{and} \quad y''(.5) = O(1), \tag{4}$$

as $\varepsilon \downarrow 0$. In a correctly scaled problem, the variable terms should be $O(1)$. Thus (3) and (4) suggest that (P) needs rescaling for $t \ll 1$, but not for $t = O(1)$.

4. **The Outer Approximation**: For $t = O(1)$, *the outer region*, we need not rescale. We will thus approximate y there using the regular perturbation exapnsion (1), along with the boundary condition $y(1) = 1$. This yields the leading-order problem

$$\begin{cases} y_0' + y_0 = 0, & \text{for } t = O(1), \\ y_0(1) = 1. \end{cases}$$

The solution is $y_0 = e^{1-t}$. We take this as our *outer approximation* $y_o(t)$. Thus

$$y(t) \approx y_o(t) = e^{1-t} \quad \text{for } t = O(1).$$

5. Definition: The region near $t = 0$ in which y is changing rapidly is called the boundary layer.

6. Balancing: Let $\delta(\varepsilon)$ be the width of the boundary layer. It is natural to rescale the problem near $t = 0$ by setting

$$\tau = \frac{t}{\delta(\varepsilon)} \quad \text{and} \quad Y(\tau) = y(t). \tag{5}$$

The new equation is

$$\frac{\varepsilon}{\delta(\varepsilon)^2} Y''(\tau) + \left[\frac{1+\varepsilon}{\delta(\varepsilon)}\right] Y'(\tau) + Y(\tau) = 0. \tag{6}$$

If the equation has been correctly rescaled for t in the boundary layer, then Y and its derivatives should be $O(1)$, with the maginitudes of the terms given by the coefficients

$$\frac{\varepsilon}{\delta(\varepsilon)^2}, \quad \frac{1}{\delta(\varepsilon)}, \quad \frac{\varepsilon}{\delta(\varepsilon)} \quad \text{and} \quad 1.$$

We determine $\delta(\varepsilon)$ by seeking a two-term dominant balance that will allow us to simplify equation (6). The simplified equation should yield an approximation to $Y(\tau)$ that satisfies the boundary condition at $\tau = 0$ as well as a *matching* condition at the right edge of the boundary layer. If it is to meet both requirements, the approximation must be the solution to a second- order equation. For this reason, one of the terms in the dominant balance must be $\varepsilon/\delta(\varepsilon)^2$. We thus have:

a. $\frac{\varepsilon}{\delta(\varepsilon)^2}$ and $\frac{1}{\delta(\varepsilon)}$ are dominant. This yields $\delta(\varepsilon) = O(\varepsilon)$, so that the dominant terms are $O(\varepsilon^{-1})$ and the others $O(1)$.

b. $\frac{\varepsilon}{\delta(\varepsilon)^2}$ and 1 are dominant. Thus $\delta(\varepsilon) = O(\sqrt{\varepsilon})$ so that the dominant terms are $O(1)$ and the "neglible" ones $O(\varepsilon^{-\frac{1}{2}})$. Clearly (b) won't work.

c. $\frac{\varepsilon}{\delta(\varepsilon)^2}$ and $\frac{\varepsilon}{\delta(\varepsilon)}$ are dominant. Thus $\delta(\varepsilon) = O(1)$, which gives dominant terms of $O(\varepsilon)$ and "negligible" ones of $O(1)$. This clearly won't work either.

We conclude that the two-term dominant balance is given by (a). We should thus take

$$\delta(\varepsilon) = \varepsilon. \tag{7}$$

Note for t in the boundary layer, $\tau = O(1)$.

7. The Inner Approximation: The choice (7) leads to the inner problem

$$\begin{cases} Y'' + Y' + \varepsilon(Y' + Y) = 0, & \text{for } \tau = O(1), \\ Y(0) = 0. \end{cases}$$

Since the scaling is now assumed correct, we may resort to regular perturbation. Set

$$Y = Y_0 + \varepsilon Y + \cdots.$$

This gives the leading-order problem

$$\begin{cases} Y_0'' + Y_0' = 0, & \text{for } \tau = O(1), \\ Y_0(0) = 0. \end{cases}$$

Thus

$$Y_0(\tau) = C_1(1 - e^{-\tau}) = C_1(1 - e^{-\frac{t}{\varepsilon}}).$$

We thus obtain the inner approximation,

$$y(t) \approx y_i(t) = C_1(1 - e^{-\frac{t}{\varepsilon}}) \quad \text{for } t = O(\varepsilon).$$

8. **Matching**: We choose C_1 to make $y_i(t)$ and $y_o(t)$ coincide (as $\varepsilon \downarrow 0$) in some intermediate zone between the boundary layer $t = O(\varepsilon)$ and the outer region $t = O(1)$. Suppose that we're in the intermediate region when

$$t = O(\alpha), \tag{8}$$

for some $\alpha = \alpha(\varepsilon)$ that is much larger than $O(\varepsilon)$ but much smaller than $O(1)$. Accordingly, we stipulate that

$$\lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\alpha(\varepsilon)} = \lim_{\varepsilon \downarrow 0} \alpha(\varepsilon) = 0. \tag{9}$$

An obvious choice is

$$\alpha(\varepsilon) = \varepsilon^{\frac{1}{2}}, \tag{10}$$

though ε^b for any $b \in (0, 1)$ would do just as well. Define the intermediate variable

$$\eta = \frac{t}{\alpha}, \tag{11}$$

which is $O(1)$ in the intermediate region. Thus

$$y_i(t) = y_i(\sqrt{\varepsilon}\eta) = C_1(1 - e^{-\frac{\eta}{\sqrt{\varepsilon}}}),$$

and

$$y_o(t) = y_o(\sqrt{\varepsilon}\eta) = e^{1 - \sqrt{\varepsilon}\eta}.$$

In order to make the approximations coincide in the intermediate region as $\varepsilon \downarrow 0$, we impose the *matching condition*:

$$\lim_{\varepsilon \downarrow 0} y_i(\sqrt{\varepsilon}\eta) = \lim_{\varepsilon \downarrow 0} y_o(\sqrt{\varepsilon}\eta),$$

so that

$$
\begin{aligned}
C_1 &= \lim_{\varepsilon\downarrow 0} C_1(1-e^{-\frac{\eta}{\sqrt{\varepsilon}}}) \\
&= \lim_{\varepsilon\downarrow 0} e^{1-\sqrt{\varepsilon}\eta} \\
&= e. \qquad (12)
\end{aligned}
$$

The inner approximation is thus

$$
y_i(t) = e(1-e^{-\frac{t}{\varepsilon}}) \quad \text{for } t = O(\varepsilon).
$$

9. **Uniform Approximation**: To obtain an approximation y_u that is valid uniformly on $[0,1]$, we add the inner and outer appoximations and subtract their common limit (8) in the intermediate zone:

$$
y_u(t) = y_i(t) + y_o(t) - e = e^{1-t} - e^{1-\frac{t}{\varepsilon}}. \qquad (13)
$$

It is easy to show that

$$
\begin{cases}
\varepsilon y_u'' + (1+\varepsilon)y_u' + y_u = 0, & \text{for } 0 < t < 1, \\
y_u(0) = 0, \\
y_u(1) = 1 - e^{1-\frac{1}{\varepsilon}}.
\end{cases}
$$

Thus y_u satisfies the differential equation and the left boundary condition of (P). At the right-hand boundary we have

$$
1 - y_u(1) = e^{1-\frac{1}{\varepsilon}} = o(\varepsilon^n) \quad \text{as } \varepsilon \downarrow 0,
$$

for any positive integer n.

10. **Note**: It isn't necessary that a uniform approximation y_u satisfy the same differential equation as the true solution y, though that does happen in the foregoing example. All we really require is that $y_u \to y$ as $\varepsilon \downarrow 0$, uniformly on the interval of interest.

11. **Example**: Consider the boundary value problem

$$
(Q)\begin{cases}
\varepsilon y'' + y' = 2t, & \text{for } 0 < t < 1,\ \varepsilon \ll 1, \\
y(0) = 1, \\
y(1) = 1.
\end{cases}
$$

Regular perturbation yields the leading order problem

$$
\begin{cases}
y_0' = 2t, & \text{for } 0 < t < 1, \\
y_0(0) = 1, \\
y_0(1) = 1.
\end{cases}
$$

The general solution to the equation is $y_0(t) = t^2 + C$. We thus cannot satisfy both boundary conditions.

12. The outer approximation is the solution to the boundary value problem

$$\begin{cases} y_0' = 2t, & \text{for } t = O(1), \\ y_0(1) = 1. \end{cases}$$

Hence

$$y_o(t) = t^2 \quad \text{for } t = O(1).$$

13. We have to determine the width $\delta(\varepsilon)$ of the boundary layer near $t = 0$. As in the previous example, we set

$$\tau = \frac{t}{\delta(\varepsilon)} \quad \text{and} \quad Y(\tau) = y(t). \tag{14}$$

The new equation is

$$\frac{\varepsilon}{\delta(\varepsilon)^2} Y''(\tau) + \frac{1}{\delta(\varepsilon)} Y'(\tau) - 2\delta(\varepsilon)\tau = 0. \tag{15}$$

The two-term dominant balance is provided by the coefficients $\varepsilon/\delta(\varepsilon)^2$ and $1/\delta(\varepsilon)$. We conclude that the boundary layer has width $\delta(\varepsilon) = O(\varepsilon)$.

14. We set $\delta(\varepsilon) = \varepsilon$ and define the new variables

$$\tau = \frac{t}{\varepsilon}, \quad \text{and} \quad Y(\tau) = y(t).$$

Then for $\tau = O(1)$,

$$\frac{1}{\varepsilon} Y'' + \frac{1}{\varepsilon} Y' - 2\varepsilon^2 \tau = 0.$$

Regular perturbation yields the leading order problem

$$\begin{cases} Y_0'' + Y_0' = 0, & \text{for } \tau = O(1), \\ Y_0(0) = 1, \end{cases}$$

with solution

$$Y_0(\tau) = (1 - C) + Ce^{-\tau}.$$

The inner approximation is thus,

$$y_i(t) = (1 - C) + Ce^{-\frac{t}{\varepsilon}} \quad \text{for } t = O(\varepsilon).$$

15. To obtain C we match in the intermediate region $t = O(\sqrt{\varepsilon})$. Define the intermediate variable

$$\eta = \frac{t}{\sqrt{\varepsilon}},$$

and stipulate that

$$\lim_{\varepsilon \downarrow 0} y_i(\sqrt{\varepsilon}\eta) = \lim_{\varepsilon \downarrow 0} y_o(\sqrt{\varepsilon}\eta).$$

Thus

$$\begin{aligned} 1 - C &= \lim_{\varepsilon \downarrow 0} (1 - C) + Ce^{-\frac{\eta}{\sqrt{\varepsilon}}} \\ &= \lim_{\varepsilon \downarrow 0} \varepsilon \eta^2 \\ &= 0. \end{aligned} \tag{16}$$

Hence $C = 1$ and

$$y_i(t) = e^{-\frac{t}{\varepsilon}} \quad \text{for } t = O(\varepsilon).$$

16. To obtain a uniformly valid approximation, add the inner and outer solutions and subtract the common intermediate limit (12):

$$y_u(t) = y_i(t) + y_o(t) = t^2 + e^{-\frac{t}{\varepsilon}}.$$

It is easily checked that

$$\varepsilon y_u'' + y_u' = 2t + 2\varepsilon.$$

Hence y_u satisfies the differential equation in (Q) as $\varepsilon \downarrow 0$. And at the boundaries,

$$y_u(0) = 1,$$

and

$$y_u(1) = 1 + e^{-\frac{1}{\varepsilon}}.$$

Thus $y_u(1) - 1 = o(\varepsilon^n)$ as $\varepsilon \downarrow 0$ for all positive n.

17. A couple of points concerning boundary layers:

a. When in doubt, assume the existence of a boundary layer at $t = 0$. If the assumption is incorrect, the procedure will break down when you try to match the inner and outer solutions in the intermediate region. At this point you may assume that there is a boundary layer near the right endpoint t_0. The analysis is the same, except that the scale transformation in the boundary layer is

$$\tau = \frac{t_0 - t}{\delta(\varepsilon)}.$$

b. It is not necessarily the case that the boundary layer has width $\delta(\varepsilon) = O(\varepsilon)$.

18. **Proposition**: Let $p(t)$ and $q(t)$ be continuous, with $p(t) > 0$ on $[0, 1]$. For the boundary value problem

$$\begin{cases} \varepsilon y'' + p(t)y' + q(t)y = 0, & \text{for } 0 < t < 1,\ \varepsilon \ll 1, \\ y(0) = a, \\ y(1) = b, \end{cases}$$

there exists a boundary layer at $t = 0$ with inner and outer approximations given respectively by

$$y_i(t) = C_1 + (a - C_1)e^{-\frac{p(0)}{\varepsilon}},$$

and

$$y_o(t) = be^{\int_t^1 \frac{q(s)}{p(s)}\,ds},$$

where

$$C_1 = be^{\int_0^1 \frac{q(s)}{p(s)}\,ds}.$$