Homework Assignment 2, Problem 17a

Let $\varphi = \varphi(x, \varepsilon)$ be a smooth function such that

$$\varphi(x_0, 0) = 0.$$

Find a perturbation expansion up to $O(\varepsilon^2)$ for the solution x to the equation

$$\varphi(x, \varepsilon) = 0,$$

where $\varepsilon \ll 1$.

Method 1
Start with the Taylor expansion of $\varphi(x, \varepsilon)$ about $(x_0, 0)$:

$$\begin{aligned} 0 &= \varphi(x, \varepsilon) \\ &= \varphi(x_0, 0) + \varphi_x(x_0, 0)(x - x_0) + \varphi_\varepsilon(x_0, 0)\varepsilon \\ &+ \frac{1}{2}\varphi_{xx}(x_0, 0)(x - x_0)^2 + \varphi_{x\varepsilon}(x_0, 0)(x - x_0)\varepsilon + \frac{1}{2}\varphi_{\varepsilon\varepsilon}(x_0, 0)\varepsilon^2 + R(x, \varepsilon), \end{aligned} \tag{1}$$

where $R(x, \varepsilon)$ is the remainder. Drop the remainder, set

$$x = x_0 + x_1\varepsilon + x_2\varepsilon^2 + O(\varepsilon^3),$$

and

$$\varphi(x_0, 0) = 0,$$

in (1). We get

$$\begin{aligned} 0 &= \varphi_x(x_0, 0)(x_1\varepsilon + x_2\varepsilon^2) + \varphi_\varepsilon(x_0, 0)\varepsilon \\ &+ \frac{1}{2}\varphi_{xx}(x_0, 0)(x_1\varepsilon + x_2\varepsilon^2)^2 + \varphi_{x\varepsilon}(x_0, 0)(x_1\varepsilon + x_2\varepsilon^2)\varepsilon + \frac{1}{2}\varphi_{\varepsilon\varepsilon}(x_0, 0)\varepsilon^2 + O(\varepsilon^3) \\ &= [\varphi_x(x_0, 0)x_1 + \varphi_\varepsilon(x_0, 0)]\varepsilon \\ &+ \left[\varphi_x(x_0, 0)x_2 + \frac{1}{2}\varphi_{xx}(x_0, 0)x_1^2 + \varphi_{x\varepsilon}(x_0, 0)x_1 + \frac{1}{2}\varphi_{\varepsilon\varepsilon}(x_0, 0)\right]\varepsilon^2 \\ &+ O(\varepsilon^3). \end{aligned}$$

Match powers of ε. We get the equations

$O(\varepsilon)$: $$\varphi_x(x_0, 0)x_1 + \varphi_\varepsilon(x_0, 0) = 0,$$

and

$O(\varepsilon^2)$: $$\varphi_x(x_0, 0)x_2 + \frac{1}{2}\varphi_{xx}(x_0, 0)x_1^2 + \varphi_{x\varepsilon}(x_0, 0)x_1 + \frac{1}{2}\varphi_{\varepsilon\varepsilon}(x_0, 0) = 0.$$

From these equations we see that the *solvability condition* is

$$\varphi_x(x_0, 0) \neq 0. \tag{2}$$

From the $O(\varepsilon)$ equation,

$$x_1 = -\frac{\varphi_\varepsilon(x_0, 0)}{\varphi_x(x_0, 0)}. \tag{3}$$

Plug this into the $O(\varepsilon^2)$ equation and solve for

$$x_2 = -\frac{\varphi_{xx}(x_0, 0)x_1^2 + 2\varphi_{x\varepsilon}(x_0, 0)x_1 + \varphi_{\varepsilon\varepsilon}(x_0, 0)}{2\varphi_x(x_0, 0)}, \tag{4}$$

where x_1 is given by (3).

Method 2
The Implicit Function theorem asserts that if condition (2) is met then the equation

$$\varphi(x, \varepsilon) = 0$$

defines a function $x = x(\varepsilon)$ whose graph (in the $x\varepsilon$-plane) passes through the point $(x_0, 0)$. The perturbation expansion

$$x = x_0 + x_1\varepsilon + x_2\varepsilon^2 + O(\varepsilon^3),$$

is simply the second-degree Taylor polynomial (with remainder) for $x(\varepsilon)$ about $\varepsilon = 0$. Thus

$$x_1 = x'(0) \quad \text{and} \quad x_2 = \frac{1}{2}x''(0).$$

To compute x_1, differentiate the equation

$$\varphi(x(\varepsilon), \varepsilon) = 0$$

with respect to ε and solve for x':

$$x'(\varepsilon) = -\frac{\varphi_\epsilon(x(\varepsilon), \varepsilon)}{\varphi_x(x(\varepsilon), \varepsilon)}. \tag{5}$$

Now set $\varepsilon = 0$ to get

$$x_1 = x'(0) = -\frac{\varphi_\epsilon(x_0, 0)}{\varphi_x(x_0, 0)}, \tag{6}$$

which agrees with (3). To complete the solution, differentiate (5) with respect to ε and set $\varepsilon = 0$. We get

$$x_2 = \frac{1}{2}x''(0) = -\frac{\varphi_{xx}(x_0, 0)x_1^2 + 2\varphi_{x\varepsilon}(x_0, 0)x_1 + \varphi_{\varepsilon\varepsilon}(x_0, 0)}{2\varphi_x(x_0, 0)}, \tag{7}$$

where x_1 is given by (6). This agrees with (4). Note that in both approaches to the problem, the solvabilty condition on the derivatives is

$$\varphi_x(x_0, 0) \neq 0.$$