Applied Math (842-843) Qualifying Exam **June 2004**

Instructions: You are to work three problems from each part below for a total of six problems. If you attempt more than six problems, clearly indicate which you wish to be graded. All problems have equal value. If you have questions about the wording of a particular problem, ask for clarification. In no case should you interpret a problem in such a way that its solution becomes trivial. Time: 3 hours.

Part I

1. Consider the following differential-integral equation that governs a population $u = u(t)$, $t > 0$:

$$\varepsilon u' = u - u^2 - u \int_0^t u(x)dx, \quad u(0) = u_0 < 1, \quad 0 < \varepsilon << 1. \tag{1}$$

This equation models logistic growth with an integral term representing the cummulative effect of a toxin on the population. You will need to know that the solution to the logistics equation $p' = p(1-p)$ is

$$p(t) = \frac{1}{1 + (1/p(0) - 1)e^{-t}}.$$

The goal of this problem is to find a uniform approximation to the solution of (1) with a "layer" near $t = 0$ using singular perturbation methods.

a. Find the outer solution.

b. Find an inner approximation using the scaling $\tau = t/\varepsilon$.

c. Find a uniformly valid approximation for $t > 0$.

2. Let f and g be smooth, real-valued functions on the interval $[a, b]$, and

$$I(\lambda) = \int_a^b f(x)e^{-\lambda g(x)}\, dx,$$

for λ real.

a. Suppose that g has a strict minimum over the interval at $c \in (a, b)$, and that $g''(c) > 0$. Sketch the derivation of the leading order Laplace asymptotic formula

$$I(\lambda) \sim f(c)\, e^{-\lambda g(c)} \sqrt{\frac{2\pi}{\lambda g''(c)}} \quad \text{as } \lambda \to \infty. \tag{2}$$

b. How would (2) have to be modified if g were strictly increasing on $(a, b]$, with $g'(a) = 0$ and $g''(a) > 0$?

3. Does the boundary value problem

$$\begin{cases} y'' - 3y' + 2y = f(x), & \text{for } 0<x<1, \\ y'(0) = 0, \\ y(1) = 0. \end{cases}$$

have a Green's function? Find it, or explain how you know it doesn't exist.

4. Solve the initial value problem

$$\begin{cases} v_t = v_{xx} + v_x, & x \in \mathbb{R} \text{ and t>0}, \\ v(x,0) = f(x). \end{cases}$$

Assume whatever you need to about the regularity and decay of f.

Part II

5. Let $u(x,t)$ be the density of a gas in a narrow tube at the point $x \in [0,1]$ and time $t > 0$.

a. Suppose that the gas density changes by diffusion only, with constant diffusion coefficient D, and that the tube is sealed at both ends. Given the initial density $u(x,0) = f(x)$, derive an initial-boundary value problem for u.

b. Solve the problem formulated in part (a). Give the solution in the form

$$u(x,t) = \int_0^1 K(x,\xi,t) f(\xi)\, d\xi,$$

for some kernel K.

6. Sketch the derivation of D'Alembert's solution,

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(z)\, dz,$$

to the initial value problem for the wave equation in one space dimension,

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & x \in \mathbb{R} \text{ and } t > 0, \\ u(x,0) = f(x), \\ u_t(x,0) = g(x). \end{cases}$$

Assume what you like about the regularity and decay of f and g.

7. Solve the initial value problem

$$\begin{cases} u_t + u_x = u^2, & x \in \mathbb{R} \text{ and } t > 0, \\ u(x,0) = g(x). \end{cases}$$

8. Let $\mathbf{x}(t,\mathbf{h})$ be the position at time t of the fluid particle initially at $\mathbf{h}$. Let $\mathbf{v} = \mathbf{v}(\mathbf{x},t)$ be the velocity field.

a. Let f be a smooth function of $\mathbf{x}$ and t, and $\Omega(t)$ a material volume. Sketch a proof of the convection theorem:

$$\frac{d}{dt} \int_{\Omega(t)} f(\mathbf{x},t)\, d\mathbf{x} = \int_{\Omega(t)} \left[\frac{Df}{Dt} + f \operatorname{div} \mathbf{v} \right] d\mathbf{x}.$$

b. Let $\varrho(\mathbf{x},t)$ be the fluid density. Prove that if mass is conserved, then

$$\frac{d}{dt} \int_{\Omega(t)} \varrho f \, d\mathbf{x} = \int_{\Omega(t)} \varrho \frac{Df}{Dt}\, d\mathbf{x}.$$