Applied Math (842-843) Qualifying Exam **January 2004**

Instructions: You are to work three problems from each part below for a total of six problems. If you attempt more than six problems, clearly indicate which you wish to be graded. All problems have equal value. If you have questions about the wording of a particular problem, ask for clarification. In no case should you interpret a problem in such a way that its solution becomes trivial. Time: 3 hours.

Part I

1. Consider a perfect gas in equilibrium with energy per unit mass e, temperature T and Boltzmann constant k. The dimensions of k are

$$[k] = \text{Energy} \times \text{Temperature}^{-1} \times \text{Mass}^{-1}.$$

Derive a functional relationship of the form $e = f(k, T)$. You may assume a unit-free law relating e, T and k.

2. Consider the boundary value problem

$$\epsilon y'' + 2xy' - 2xy = 0,\ 0 < x < 1,\ 0 < \epsilon \ll 1,$$

with boundary conditions $y(0) = 0$ and $y(1) = e$. Use the method of matched asymptotic expansions to determine a uniformly valid leading order approximation for the solution of this problem. (Your answer may involve the erf function $w(x) = \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds$, which satisfies $w(\infty) = 1$.)

3. Find the extremals of the functional

$$J(y) = \int_0^1 \left(y^2 + y'y + (y' - 2)^2 \right) dx$$

over the domain $A = \left\{ y \in C^2[0,1] \,|\, y(0) = 0 = y(1) \right\}$. Show that J does not assume a maximum value at these extremals. Explain why it does or does not follow that they yield minimum values of J.

4. Consider the nonlinear system

$$\begin{aligned} x' &= 1 - (p+1)\, x + x^2 y \\ y' &= px - x^2 y \end{aligned}$$

where $p > 0$ is a parameter. Find all equilibrium solutions and determine for what value(s) of p a bifurcation occurs. Also discuss stability of the equilibria for different values of p.

Part II

5. Derive the Green's function $g(x, \xi)$ for the regular Sturm-Liouville problem $-u'' = f(x)$, $0 < x < 1$ with boundary conditions $u(0) = 0$ and $u'(1) = 0$.

6. Consider the Dirichlet problem on the upper half-plane:

$$\begin{aligned} u_{xx} + u_{yy} &= 0,\ x \in \mathbb{R} \text{ and } y > 0, \\ u(x, 0) &= f(x),\ x \in \mathbb{R}. \end{aligned}$$

Assume anything you like about the regularity and decay of f, and use the Fourier transform to derive a representation for *bounded* solutions $u(x,y)$ as

$$u(x,y) = \int_{-\infty}^{\infty} P(x-\xi, y) f(\xi)\, d\xi.$$

Give $P(x,y)$ explicitly.

7. Consider two conservation laws for a nonnegative function $u(x,t)$ representing some physical quantity, namely $u_t + \left(\frac{1}{2}u^2\right)_x = 0$ and $\left(u^2\right)_t + \left(\frac{2}{3}u^3\right)_x = 0$, where $x \in \mathbb{R}$ and $t > 0$, together with an initial condition $u(x,0) = \phi(x)$.
(a) Show that these conservation laws have the same classical (nonnegative) solutions $u(x,t)$.
(b) Show that the two conservation laws of (a) need not have the same weak solutions (Hint: consider the initial condition $u(x,0) = 1$ for $x \le 0$ and $u(x,0) = 0$ for $x > 0$.)

8. A gas exhibits a one-dimensional flow in a linear tube of constant circular cross-section with a density function $\rho(x,t)$ (mass per unit volume) and velocity function $v(x,t)$. From first principles, express the conservation of mass as a partial differential equation.