Integral Asymptotics: Laplace's Method

1. We'll use Laplace's method to determine the leading-order behavior of the integal

$$I(\lambda) = \int_a^b f(t) e^{-\lambda g(t)}\, dt, \tag{1}$$

as $\lambda \to \infty$. We'll assume without further comment that $I(\lambda)$ converges for λ sufficiently large, that f and g are smooth enough near to be replaced by local Taylor approximations of appropriate degree.

2. We'll first consider the case in which g assumes a strict minimum over $[a, b]$ at an interior critical point c. Assume that

- $g'(c) = 0$,
- $g''(c) > 0$,
- $f(c) \neq 0$.

We can rewrite the integral as

$$I(\lambda) = e^{-\lambda g(c)} \int_a^b f(t) e^{-\lambda[g(t)-g(c)]}\, dt. \tag{2}$$

The main idea is this: For $\lambda \gg 1$, the main contribution to the integral comes from a small neighborhood of c. Thus, for $\lambda \gg 1$,

$$\begin{aligned}
I(\lambda) &\approx e^{\lambda g(c)} \int_{c-\varepsilon}^{c+\varepsilon} f(t) e^{-\lambda[g(t)-g(c)]}\, dt \\
&\approx e^{-\lambda g(c)} f(c) \int_{c-\varepsilon}^{c+\varepsilon} e^{-\lambda[g(t)-g(c)]}\, dt \\
&\approx e^{-\lambda g(c)} f(c) \int_{c-\varepsilon}^{c+\varepsilon} e^{-\lambda[g'(c)(t-c)+\frac{1}{2}g''(c)(t-c)^2]}\, dt \\
&= e^{-\lambda g(c)} f(c) \int_{c-\varepsilon}^{c+\varepsilon} e^{-\frac{\lambda}{2}g''(c)(t-c)^2}\, dt \\
&\approx e^{-\lambda g(c)} f(c) \int_{-\infty}^{\infty} e^{-\frac{\lambda}{2}g''(c)(t-c)^2}\, dt \\
&= e^{-\lambda g(c)} f(c) \int_{-\infty}^{\infty} e^{-\frac{\lambda}{2}g''(c)s^2}\, ds \\
&= e^{-\lambda g(c)} f(c) \sqrt{\frac{2\pi}{\lambda g''(c)}}.
\end{aligned}$$

Thus, to leading order,

$$I(\lambda) \sim e^{-\lambda g(c)} f(c) \sqrt{\frac{2\pi}{\lambda g''(c)}} \quad \text{as } \lambda \to \infty. \tag{3}$$

Here, the symbol "$\sim$" means that the right-hand side is the first term in an asymptotic expansion of the left-hand side.

3. If g has its minimum over $[a, b]$ at an endpoint (say, $t = a$) with $g'(a) = 0$, $g''(a) > 0$, then analysis similar to the foregoing yields

$$I(\lambda) \sim e^{-\lambda g(a)} f(a) \sqrt{\frac{\pi}{2\lambda g''(a)}} \quad \text{as } \lambda \to \infty, \tag{4}$$

with the obvious modification when $t = b$.

4. **Example**: We can use the method of Laplace to determine the leading order behavior of

$$I(\lambda) = \int_{-1}^{1} \frac{\sin t}{t} e^{-\lambda \cosh t}\, dt.$$

Let $g(t) = \cosh t$ and $f(t) = \sin t/t$. The function g assumes a strict minimum over $[-1, 1]$ at the interior point $t = 0$, with $g'(0) = 0$ and $g''(0) = 1$. And since $f(0) = 1$, we have by (3),

$$I(\lambda) \sim e^{-\lambda} \sqrt{\frac{2\pi}{\lambda}} \quad \text{as } \lambda \to \infty. \tag{5}$$

5. There are three ideas behind Laplace's method. These are

a. For $\lambda \gg 1$, the main contribution to $I(\lambda)$ comes from a small region of the minimizer $t = c$. We can thus replace an integral over $[a, b]$ with an integral over $(c - \varepsilon, c + \varepsilon)$. (Or over $[a, a + \varepsilon)$ or $(b - \varepsilon, b]$ as the case may be).

b. In the small neighborhood of the minimizer, we can approximate $f(t)$ and $g(t)$ with Taylor polynomials.

c. We may extend the interval of integration to include any region that only contributes higher-order terms to $I(\lambda)$ as $\lambda \to \infty$.

6. **Note**: Formulas (3) and (4) are valid for infinite and semi-infinite intervals of integration, provided $I(\lambda)$ converge for λ large.

7. **The Gamma Function:** For $x > 0$ we define the gamma function

$$\Gamma(x) = \int_{0}^{\infty} t^{x-1} e^{-t}\, dt.$$

It isn't hard to show that

$$\Gamma(1) = 1 \quad \text{and} \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

To prove the second claim we make the change of variable $t^{\frac{1}{2}} = u$ and use the fact that

$$\int_0^\infty e^{-au^2}\, du = \frac{1}{2}\sqrt{\frac{\pi}{a}},$$

for $a > 0$.

8. Integration by parts yields

$$\Gamma(x+1) = x\Gamma(x), \tag{6}$$

for any $x > 0$. Thus $\Gamma(2) = \Gamma(1) = 1$, $\Gamma(3) = 2\Gamma(2) = 2 \times 1$, $\Gamma(4) = 3\Gamma(3) = 3 \times 2 \times 1$, etc. In general,

$$\Gamma(n+1) = n!,$$

for any nonnegative integer n. Thus (6) tells us that the gamma function is a continuous generalization of the factorial function.

9. **Example**: Derive Stirling's approximation:

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}, \quad \text{as } n \to \infty.$$

$$\begin{aligned}
\Gamma(x+1) &= \int_0^\infty t^x e^{-t}\, dt \\
&= \int_0^\infty e^{x \ln t} e^{-t}\, dt \\
&= \int_0^\infty e^{-x\left(\frac{t}{x} - \ln t\right)}\, dt \quad (\text{Set } t = xz.) \\
&= x \int_0^\infty e^{-x\left(z - \ln xz\right)}\, dz \\
&= x e^{x \ln x} \int_0^\infty e^{-x\left(z - \ln z\right)}\, dz \\
&= x^{x+1} \int_0^\infty e^{-x\left(z - \ln z\right)}\, dz. \qquad (7)
\end{aligned}$$

Apply Laplace's method to the integral in (7) with $f(z) \equiv 1$ and $g(z) = z - \ln z$. We see that g has a strict minimum over $(0, \infty)$ at $z = 1$, with $g(1) = 1$, $g'(1) = 0$ and $g''(1) = 1$. Thus,

$$\int_0^\infty e^{-x\left(z - \ln z\right)}\, dz \sim \sqrt{\frac{2\pi}{x}} e^{-x}. \quad \text{as } x \to \infty,$$

and so

$$\Gamma(x+1) \sim x^{x+1}\sqrt{\frac{2\pi}{x}}e^{-x} = \sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x} \quad \text{as } x \to \infty.$$

Now set $x = n$ and use the fact that $\Gamma(n+1) = n!$.

10. **Higher-Order Asymptotics:** With the gamma function, we can find higher-order terms in an asymptotic expansion. Suppose for example that $g(t)$ assumes a strict minimum over $[a, b]$ at an interior point c, that $g'(c) = 0$, $g''(c) \neq 0$, $f(c) = 0$ and $f''(c) \neq 0$. Then for $\lambda \gg 1$,

$$\begin{aligned}
I(\lambda) &\approx e^{\lambda g(c)} \int_{c-\varepsilon}^{c+\varepsilon} f(t)e^{-\lambda[g(t)-g(c)]}\,dt \\
&\approx e^{-\lambda g(c)} \int_{c-\varepsilon}^{c+\varepsilon} \left[f'(c)(t-c) + \frac{f''(c)}{2}(t-c)^2\right] e^{-\frac{\lambda}{2}g''(c)(t-c)^2}\,dt \\
&= e^{-\lambda g(c)} \int_{-\infty}^{\infty} \left[f'(c)s + \frac{f''(c)}{2}s^2\right] e^{-\frac{\lambda}{2}g''(c)s^2}\,dt \\
&= \frac{f''(c)}{2}e^{-\lambda g(c)} \int_{-\infty}^{\infty} s^2 e^{-\frac{\lambda}{2}g''(c)s^2}\,dt \\
&= f''(c)e^{-\lambda g(c)} \int_0^{\infty} s^2 e^{-\frac{\lambda}{2}g''(c)s^2}\,dt.
\end{aligned}$$

Make the change of variable

$$\frac{\lambda}{2}g''(c)s^2 = u,$$

and use properties of the gamma function to show that

$$\int_0^{\infty} s^2 e^{-\frac{\lambda}{2}g''(c)s^2}\,dt = \sqrt{\frac{\pi}{(\lambda g''(c))^3}}.$$

Thus, to leading order,

$$I(\lambda) \sim f''(c)e^{-\lambda g(c)}\sqrt{\frac{\pi}{(\lambda g''(c))^3}}, \quad \text{as } \lambda \to \infty.$$