Green's Functions

1. The Dirac δ function is defined on $\mathbf{R}^n$ by

$$\delta(x) = \begin{cases} 0 & \text{for } x \neq 0, \\ \infty & \text{for } x = 0, \end{cases} \tag{1}$$

with

$$\int_{\mathbf{R}^n} \delta(x)\,dx = 1. \tag{2}$$

From properties (1) and (2), we can show that

$$\int_B \delta(x)\,dx = \begin{cases} 0 & \text{if } x \notin B, \\ 1 & \text{if } x \in B, \end{cases} \tag{3}$$

and that

$$\int_B \delta(x) f(x)\,dx = \begin{cases} 0 & \text{if } 0 \notin B, \\ f(x) & \text{if } 0 \in B, \end{cases} \tag{4}$$

This last easily generalizes to

$$\int_B \delta(z-x) f(x)\,dx = \begin{cases} 0 & \text{if } z \notin B, \\ f(z) & \text{if } z \in B, \end{cases} \tag{5}$$

Thus,

$$\int_{\mathbf{R}^n} \delta(z-x) f(x)\,dx = f(z). \tag{6}$$

2. There are two problems: The first is that the δ function cannot exist in the traditional sense of the word "function." The second is that the judicious use of the δ function always yields the right answer. We can solve both these problems by defining the δ function as a generalized functio or distribution.

3. Let L be a linear differential operator. For example,

$$L = \sum_{i=0}^{n} a_n(x) \frac{d^n}{dx^n}, \tag{7}$$

or a partial differential operator,

$$L = \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}, \tag{8}$$

which is the n-dimensional Laplacian. To solve the inhomogeneous equation

$$Lu(x) = f(x),$$

we assume that L can be inverted and that the inverse takes the form of an integral operator:

$$u(x) = L^{-1} f(x) = \int_B K(x,y) f(y)\, dy.$$

Thus

$$f(x) = Lu(x) = L \int_B K(x,y) f(y)\, dy \text{“}=\text{”} \int_B LK(x,y) f(y)\, dy. \tag{9}$$

It follows that in some sense,

$$LK(x,y) = \delta(x-y). \tag{10}$$

A function satisfying (10) is called a fundamental solution for L.

4. Since the δ function is not really a function, we will at some point have to explain (10) more carefully. What does it mean to apply a differential operator L to a function K to obtain a distribution, or generalized function δ?

5. **Vague Definition:** A Green's function is a fundamental solution that has been equipped with certain boundary values.

6. Consider the second-order, ordinary differential operator

$$L = a_2(x) D^2 + a_1(x) D + a_0(x), \tag{11}$$

where the a_i are smooth and $a_2(x) \neq 0$ on the interval I of interest. The homogeneous problem is

$$Lu = 0, \tag{12}$$

and the inhomogeneous problem,

$$Lu = f(x), \tag{13}$$

for some given function f. Take f to be continuous.

7. Let u_1 and u_2 be linearly independent solutions to the homogeneous problem, and u_p a particular solution to the inhomogeneous problem. Then the Wronskian

$$W(x) = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix}$$

is nonzero on I. The general solution to () is

$$u(x) = c_1 u_1(x) + c_2 u_2(x) + u_p(x), \tag{14}$$

where u_p a particular solution to ().

8. The variation of parameters formula gives the particular solution

$$u_p(x) = u_2(x) \int_a^x \frac{u_1(z)}{a_2(z) W(z)} f(z)\, dz - u_1(x) \int_a^x \frac{u_2(z)}{a_2(z) W(z)} f(z)\, dz. \tag{15}$$

9. Suppose that we want a fundamental solution for L. Think of y as a parameter, and set $u(x) = K(x, y)$ and $f(x) = \delta(x - y)$ in (). Then by (), every fundamental solution of L is of the form

$$K(x, y) = c_1(y)u_1(x) + c_2(y)u_2(x) + K_p(x, y), \tag{16}$$

where

$$\begin{aligned} K_p(x, y) &= u_2(x) \int_a^x \frac{u_1(z)}{a_2(z)W(z)} \delta(z - y)\, dz - u_1(x) \int_a^x \frac{u_2(z)}{a_2(z)W(z)} \delta(z - y)\, dz \\ &= \begin{cases} 0 & \text{for } x < y, \\ \frac{u_2(x)u_1(y) - u_1(x)u_2(y)}{a_2(y)W(y)} & \text{for } x \geq y. \end{cases} \end{aligned} \tag{17}$$

10. You have the right to ask whether (17) really is a fundamental solution, i.e. whether

$$L \int_a^b K(x, y) f(y)\, dy = f(x), \tag{18}$$

for f continuous on $[a, b]$. The answer is "yes" as long as we assume that c_1 and c_2 are continuous functions of y. Note that with this assumption,

a. $K(x, y)$ is continuous,

b. $K(x, y)$ is C^2 in x for $x \neq y$,

c. $LK(x, y) = 0$ for $x \neq y$, and

d. $K(x, x + 0) - K(x, x - 0) = -a_2(x)^{-1}$.

It follows from (a)-(d) that

$$L \int_a^b K(x, y) f(y)\, dy = f(x).$$

11. **Proposition:** Any function $F(x, y)$ satisfying (a)-(d) is a fundamental solution for L.

12. Let $L = D^2 + 1$. Then $u_1 = \cos x$, $u_2 = \sin x$, $a_2(x) \equiv 1$ and $W(x) \equiv 1$. Thus

$$K_p(x, y) = \begin{cases} 0 & \text{for } x < y, \\ \sin(x - y) & \text{for } x \geq y. \end{cases}$$

Every fundamental solution for L is of the form

$$K(x, y) = c_1(y) \cos x + c_2(y) \sin x + K_p(x, y). \tag{19}$$

13. Two-Point Boundary Value Problems: We often have to solve the equation $Lu(x) = f(x)$ on $I = [a, b]$ subject to boundary conditions on u and u'. Common conditions are

Homogeneous Dirichlet: $u(a) = u(b) = 0$.

Homogeneous Neumann: $u'(a) = u'(b) = 0$.

Separated Boundary Conditions: $\alpha_0 u(a) + \alpha_1 u'(a) = 0$ and $\beta_0 u(b) + \beta_1 u'(b) = 0$.

Periodic Boundary Conditions: $u(a) = u(b)$ and $u'(a) = u'(b)$.

We can use boundary operators B_1 and B_2 in the representation of boundary conditions. Note that in the examples, the boundary operators are linear. The problem

$$(P)\begin{cases} Lu = f(x) & \text{for } a < x < b, \\ B_1 u = 0, \\ B_2 u = 0, \end{cases}$$

is called a two-point boundary value problem.

14. Suppose we want to solve (P). Let $G(x, y)$ be a fundamental solution satisfying (as a function of x with $a < y < b$) the boundary conditions:

$$B_1 G(y) = B_2 G(y) = 0 \quad \text{for } a < y < b. \tag{20}$$

Let

$$u(x) = \int_a^b G(x, y) f(y)\, dy. \tag{21}$$

Since G is a fundamental solution, u really does satisfy the equation $Lu = f$. As for the boundary conditions,

$$B_1 u = B_1 \int_a^b G(x, y) f(y)\, dy = \int_a^b B_1 G(y) f(y)\, dy = 0, \tag{22}$$

and

$$B_2 u = B_2 \int_a^b G(x, y) f(y)\, dy = \int_a^b B_2 G(y) f(y)\, dy = 0. \tag{23}$$

Hence u is a solution to (P)

15. How do we find the Green's function? Let's consider the case of homogeneous Dirichlet data:

1. Since the Green's function is a fundamental solution, it must be of the form

$$G(x, y) = c_1(y) u_1(x) + c_2(y) u_2(x) + K_p(x, y). \tag{24}$$

We already have K_p, u_1 and u_2 so it only remains to find c_1 and c_2

2. Apply the boundary operators:

$$B_1 G(y) = c_1(y) B_1 u_1 + c_2(y) B_1 u_2 + B_1 K_p(y) = 0,$$

and

$$B_2G(y) = c_1(y)B_2u_1 + c_2(y)B_2u_2 + B_2K_p(y) = 0.$$

In matrix form, this is

$$\begin{bmatrix} B_1u_1 & B_1u_2 \\ B_2u_1 & B_2u_2 \end{bmatrix} \begin{bmatrix} c_1(y) \\ c_2(y) \end{bmatrix} = - \begin{bmatrix} B_1K_p(y) \\ B_2K_p(y) \end{bmatrix}. \tag{25}$$

We can solve (25) uniquely for the $c_i(y)$ if and only if

$$\det(B_iu_j) \neq 0. \tag{26}$$

3. If (26) holds, then we solve for the coefficient functions $c_i(y)$ and for the Green's function $G(x, y)$.

16. **Example:** Consider the boundary value problem

$$(P_1) \begin{cases} u'' + u = f(x) & \text{for } 0 < x < \pi, \\ u'(0) = 0, \\ u(\pi) = 0. \end{cases}$$

Here, $L = D^2 + 1$, $B_1u = u(0)$ and $B_2u = u(\pi)$. We take as linearly independent solutions to the homogeneous problem $u_1 = \cos x$ and $u_2 = \sin x$. Since

$$\det(B_iu_j) \neq 0,$$

the Green's function exists. Following the above recipe yields

$$G(x, y) = \begin{cases} \cos x \, \sin y & \text{for } 0 \leq x < y < \pi, \\ \sin x \, \cos y & \text{for } 0 < y < x \leq 1. \end{cases}$$

Hence a solution to (P_1) is

$$u(x) = \int_0^{\pi} G(x, y) \, f(y) \, dy.$$