The Fourier Transform 1

1. The notation of Laurent Schwartz: A multi-index is an n-tuple

$$\alpha = (\alpha_1, \ldots, \alpha_n),$$

where the a_i are nonnegative integers. If a is a multi-index, then

$$|\alpha| = \alpha_1 + \cdots + \alpha_n.$$

The general partial differential monomial is

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha^n}}.$$

For example, if

$$\alpha = (1, 0, 2),$$

then

$$D^\alpha = \frac{\partial^3}{\partial x_1 \partial x_3^2}.$$

So if $u = u(x_1, x_2, x_3)$ then

$$D^\alpha u = \frac{\partial^3 u}{\partial x_1 \partial x_3^2}.$$

If $\alpha = (0, \ldots, 0)$, we set

$$D^\alpha u = u.$$

For $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$,

$$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}.$$

2. The Fourier transform of f is

$$(\mathcal{F} f)(\xi) = \hat{f}(\xi) = \int f(x) e^{-2\pi i \xi \cdot x}\, dx, \tag{1}$$

where the integral is taken over $\mathbf{R}^n$.

3. **Note**: In the study of the Fourier transform, it is necessary to integrate functions and partial derivatives of all orders over $\mathbf{R}^n$. For this reason, one generally defines the Fourier transform first for a set of infinitely smooth, rapidly decaying functions called the Schwartz class. Rather than make a careful preliminary study of the Schwartz class, we will just assume that functions are as smooth we need, and that those functions and their partial derivatives decay as rapidly as we need. This saves us any worry about differentiability and convergence of integrals.

4. Liouville's theorem (and a little complex contour integration) allows us to show that for compex numbers a and b such that $\Re a > 0$,

$$\int_{-\infty}^{\infty} e^{-a(x-b)^2}\, dx = \sqrt{\frac{\pi}{a}}.$$

5. **Example**: For $\varepsilon > 0$ and $x \in \mathbf{R}^n$, the Gauss kernel is

$$G_\varepsilon(x) = \left(\frac{1}{4\pi\varepsilon}\right)^{\frac{n}{2}} e^{-\frac{|x|^2}{4\varepsilon}}.$$

The Fourier transform is

$$\hat{G}_\varepsilon(\xi) = \int G_\varepsilon(x) e^{-2\pi i \xi \cdot x}\, dx = e^{-4\varepsilon\pi^2|\xi|^2}.$$

6. The function G_ε has some useful properties:

a. $G_\varepsilon(x) \geq 0$ for all x.

b. $\int G_\varepsilon(x)\, dx = 1$ for all $\varepsilon > 0$.

c. For $\delta > 0$,

$$\lim_{\varepsilon \downarrow 0} \int_{|x| \geq \delta} G_\varepsilon(x)\, dx = 0.$$

d. For $\delta > 0$,

$$\lim_{\varepsilon \downarrow 0} \sup_{|x| \geq \delta} G_\varepsilon(x) = 0.$$

It isn't hard to see (draw some pictures) that in some sense

$$G_\varepsilon(x) \to \delta(x) \quad \text{as} \quad \varepsilon \downarrow 0. \tag{2}$$

You can in fact show that for any function $G_\varepsilon(x)$ with properties (a)-(d),

$$\lim_{\varepsilon \downarrow 0} \int G_\varepsilon(x-y) f(y)\, dy = f(x), \tag{3}$$

as long as f is reasonably nice. Thus (2) holds in the sense of (3). The function G_ε is called an *approximate identity* as $\varepsilon \downarrow 0$.

7. **Proposition**: The Fourier transform is linear.

8. Proposition: For $a \in \mathbf{R}^n$, let τ_a be the translation operator

$$(\tau_a f)(x) = f(x-a).$$

Then

$$(\tau_a f)\hat{}(\xi) = \hat{f}(\xi) e^{-2\pi i \xi \cdot a}.$$

9. Proposition: For $a \in \mathbf{R}^n$, let μ_a be the modulation operator

$$(\mu_a f)(x) = f(x) e^{-2\pi i a \cdot x}.$$

Then

$$(\mu_a f)\hat{}(\xi) = \hat{f}(\xi + a).$$

10. Proposition: For any multi-index α,

$$(D^\alpha f)\hat{}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi) = (2\pi i)^{|\alpha|} \xi^\alpha \hat{f}(\xi).$$

To see this, use integration by parts and the rapid decay of f to show that

$$\widehat{f_{x_j}}(\xi) = 2\pi i \xi_j \hat{f}(\xi).$$

Schematically,

$$\frac{\partial}{\partial x_j} \xrightarrow{\mathcal{F}} 2\pi i \xi_j. \tag{4}$$

Repeated application of (4) yields the advertised formula.

11. Example: For $f : \mathbf{R}^n \mapsto \mathbf{C}$ suffciently smooth and rapidly decaying,

$$(\Delta f)\hat{}(\xi) = -4\pi^2 |\xi|^2 \hat{f}(\xi). \tag{5}$$

12. Definition: The convolution of f and g in is

$$(f * g)(x) = \int f(x-y) g(y)\, dy.$$

13. Proposition: $(f * g)(x) = (g * f)(x)$.

14. Proposition: $(f * g)\hat{}(\xi) = \hat{f}(\xi)\, \hat{g}(\xi)$.

15. **Definition**: The inverse Fourier transform of a function $g(\xi)$ is

$$\check{g}(x) = (\mathcal{F}^{-1}g)(x) = \int g(\xi)e^{2\pi i\xi\cdot x}\,d\xi. \tag{6}$$

Note that $\mathcal{F}^{-1}$ is also a linear operator.

16. The name "inverse Fourier transform" suggests that

$$(\hat{g})\check{}(x) = g(x). \tag{7}$$

As we showed in class, (7) is indeed true, at least for smooth functions g of comapct support on $\mathbf{R}$. It is actually true (in various senses) for much larger classes of functions defined on $\mathbf{R}^n$. You can establish some of these results using eigenfunction expansions, or with approximate identities like the Gauss kernel.

17. Define the L^2 inner product on $\mathbf{R}^n$ by

$$\langle f\,,g\rangle = \int f(x)\bar{g}(x)\,dx.$$

The corresponding L^2 norm is

$$\|f\|_2 = \sqrt{\langle f\,,f\rangle} = \left\{\int |f(x)|^2\,dx\right\}^{\frac{1}{2}}.$$

The *Plancherel identity* is

$$\langle f\,,g\rangle = \langle \hat{f}\,,\hat{g}\rangle, \tag{8}$$

for all f, g in $L^2(\mathbf{R}^n)$. If we set $g = f$ in (8), we get

$$\|f\|_2 = \|\hat{f}\|_2. \tag{9}$$

In class, we used eigenfunction expansions to prove (8) for smooth functions of compact support on $\mathbf{R}$.