Math 842 Final Exam. Do six problems. Show your work.

1. Consider the initial value problem for a pendulum of length l rotating about the vertical axis:

$$(P_0)\begin{cases} \ddot{\theta} + \left(\frac{g}{l} - \Gamma^2 \cos\theta\right) \sin\theta = 0, \\ \theta(0) = a > 0, \\ \dot{\theta}(0) = 0. \end{cases}$$

Here, Γ is a *small* constant and θ is dimensionless.

[6] **a**. Use the initial data and the fact that $[g/l] = (\text{Time})^{-2}$ to rewrite (P_0) as a problem for $\varphi(\tau)$, where φ and τ are the dimensionless angle and time respectively. The new initial angle should be $\varphi(0) = 1$.

[4] **b**. Suppose that

$$\frac{\Gamma^2 l}{ga} = \varepsilon \ll 1. \tag{1}$$

Set $\varphi = \varphi_0 + O(\varepsilon)$ and show that

$$\varphi_0'' + \frac{\sin(a\varphi_0)}{a} = 0.$$

Thus, under the condition (1), to leading order, the rotating pendulum behaves like the usual pendulum.

[10] **2**. A physical system is governed by a unit-free law of the form $f(E, P, A) = 0$, where E, P and A are energy, pressure and area respectively. Show that $PA^{\frac{3}{2}}/E = \text{constant}$.

[10] **3**. Use singular perturbation to find an approximate solution to

$$\begin{cases} \varepsilon y'' + y' = 2t, \quad \text{for } 0 < t < 1,\ \varepsilon \ll 1, \\ y(0) = 1, \\ y(1) = 1. \end{cases}$$

The approximation must be valid as $\varepsilon \downarrow 0$, uniformly on $[0, 1]$.

4. Consider the integral

$$I(\lambda) = \int_0^1 \sqrt{1+t}\, e^{-\lambda(t^2-2t)}\, dt.$$

[8] **a**. Use Laplace's method to determine the leading-order behavior of $I(\lambda)$ as $\lambda \to \infty$.

[2] **b**. Find the second nonzero term in the asymptotic expansion of $I(\lambda)$.

[10] **5**. Find the full asymptotic expansion of the integral

$$I(\lambda) = \int_0^{\frac{\pi}{2}} e^{-\lambda \sin^2 \theta} \sec\theta \, d\theta.$$

as $\lambda \to \infty$.

6. The compementary error function is

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} \, dt.$$

[5] **a**. Use integration by parts to obtain leading-order behavior of $\mathrm{erfc}(x)$ as $x \to \infty$.

[5] **b**. Use Watson's lemma to obtain the full asymptotic expansion of $\mathrm{erfc}(x)$ as $x \to \infty$.

[10] **7**. Determine the leading-order behavior of the integral

$$I(\lambda) = \int_0^{\pi} e^{\lambda i\,(2\sin t + t)}\, (1+t)^{\frac{3}{2}} \, dt,$$

as $\lambda \to \infty$.

[10] **8**. For the domain $\mathcal{D}$ of smooth functions y satisfying the boundary conditions $y(0) = 1$ and $y(\pi/2) = 2$, let

$$J(y) = \int_0^{\frac{\pi}{2}} \frac{y'(x)^2}{\cos x} dx.$$

Find the extremals of J over $\mathcal{D}$.

[10] **9**. A mechanical system has generalized coordinates and momenta $q = (q_1, \ldots, q_n)$ and $p = (p_1, \ldots, p_n)$, and Hamiltonian $H(q,p)$. The *Poisson bracket* of functions $F(q,p)$ and $G(q,p)$ is

$$\{F, G\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i} \frac{\partial F}{\partial p_i} \right).$$

Show that

$$\frac{d}{dt} F(q(t), p(t)) = \{F, H\}.$$

(Thus, if $\{F, H\} = 0$, then F is a conserved quantity.)

10. Let $x = (x_1, x_2)$ be the rectangular coordinates of a body of mass m moving under the influence of a central force field. The kinetic and potential energies are

$$T(\dot{x}) = \frac{1}{2}m|\dot{x}|^2 \quad \text{and} \quad U(x) = -\frac{c}{|x|},$$

where c is a positive constant.

[5] **a**. Introduce polar coordinates; the distance q_1 from the origin and the angle q_2. These are your generalized coordinates. Show that the Lagrangian is

$$L(q, \dot{q}) = \frac{1}{2}m(\dot{q}_1^2 + q_1^2\dot{q}_2^2) + \frac{c}{q_1}.$$

Write out the equations of motion in Lagrangian form.

[5] **b**. Let p_1 and p_2 be the generalized momenta. Give the Hamiltonian as a function of q and p. Write out Hamilton's equations. Is the total energy conserved?

11. The surface $\mathcal{S}$ given by

$$T(u, v) = \left(u, v, \frac{u^2}{2}\right),$$

is a parabolic cylinder.

[4] **a**. Write out the Euler equations for the curves $(u(t), v(t))$ whose images under T are geodesics on $\mathcal{S}$.

[3] **b**. Identify one first integral of the Euler equations.

[3] **c**. Use the first integral to show that the image of a curve $v = v(u)$ satisfying the equation

$$\frac{dv}{du} = A\sqrt{1 + u^2}, \tag{2}$$

for some constant A, is an extremal. (Hint: Pull a $\dot{u}$ out of the denominator of the first integral, then solve for $v'(u)$. Equation (2) isn't hard to solve, but let's not bother with it now.)