Problems and Equations

1. Let B be a bounded subset of $\mathbf{R}^3$ enclosed by the smooth surface Q. Let ν be the outer unit normal to Q. Let

$$\tilde{Q} = Q \times [0, \infty).$$

Thus (x, t) lies in $\tilde{Q}$ if $x \in Q$ and $t \geq 0$.

2. Let $u = u(x, t)$ be the temperature of a body occupying the region B at the point x and time t. Assume that

a. The flux density vector is given by Fourier's law with a diffusion coefficient D, and

b. That there are neither sinks nor sources in B. (Thus heat energy is conserved.)

The assumptions imply that u satisfies the heat equation

$$u_t - D\Delta u = 0. \tag{1}$$

Let the initial temperature distribution be

$$u(x, 0) = f(x), \tag{2}$$

and the temperature on the boundary at time t be

$$u\big|_{\tilde{Q}} = g(x, t). \tag{3}$$

The initial-boundary value problem (P_1) for u comprises equation (1) along with the conditions (2) and (3):

$$(P_1) \begin{cases} u_t - D\Delta u = 0, & \text{for } x \in B \text{ and } t > 0, \\ u(x, 0) = f(x), \\ u\big|_{\tilde{Q}} = g(x, t). \end{cases}$$

3. If we prescribe the heat flux across Q rather than the temperature on Q, then the boundary condition (3) is replaced by

$$D_\nu u\big|_{\tilde{Q}} = h(x, t). \tag{4}$$

We thus have the initial-boundary value problem

$$(P_2) \begin{cases} u_t - D\Delta u = 0, & \text{for } x \in B \text{ and } t > 0, \\ u(x, 0) = f(x), \\ D_\nu u\big|_{\tilde{Q}} = h(x, t). \end{cases}$$

If, for example, the body is insulated, we have the "no-flux" boundary condition

$$D_\nu u\big|_{\tilde{Q}} = 0. \tag{5}$$

4. (P_1) and (P_2) are typical initial-boundary value problems for a parabolic equation.

5. Suppose that, as $t \to \infty$, the temperature distribution achieves a time-independent equilibrium $w = w(x)$. As we've seen, w is harmonic in B:

$$\Delta w(x) = 0, \tag{6}$$

for $x \in B$. It makes no sense to assign an initial value to w, but we can prescribe w on the boundary:

$$w\big|_Q = g(x). \tag{7}$$

We thus have the Dirichlet problem

$$(P_3)\begin{cases} \Delta w = 0, & \text{for } x \in B, \\ w\big|_Q = g(x). \end{cases}$$

The boundary condition (7) is called the Dirichlet condition.

6. Suppose that there is a time-independent heat source $h(x)$ in the original model, and that instead of u, we prescribe the heat flux on Q. The long-term temperature equilibrium would then be governed by

$$(P_4)\begin{cases} -\Delta w = \varphi(x), & \text{for } x \in B, \\ D_\nu w\big|_Q = g(x), \end{cases} \tag{8}$$

where $\varphi(x) = D^{-1}h(x)$. This is the Neumann problem for Poisson's equation. The condition (8) is a Neumann condition.

7. You can combine the Dirichlet and Neumann conditions to get a mixed boundary condition

$$a(x)w(x) + b(x)D_\nu w(x)\big|_{x\in Q} = g(x).$$

8. (P_3) and (P_4) are typical boundary value problems for elliptic equations.

9. A thin, elastic membrane is stretched over a region B in $\mathbf{R}^2$ which is bounded by a smooth, closed curve Q. The membrane is clamped along Q. A disturbance causes small amplitude vibrations in the membrane. Let $u = u(x,t)$ be the displacement from equilibrium at $x \in B$ and time t. We assume that u satisfies the wave equation

$$u_{tt} - c^2\Delta u = 0, \tag{9}$$

where $c > 0$ is a constant and Δ is the 2-dimensional Laplacian. (If there were an external force driving the vibrations, it would be represented by an inhomogeneity on the right-hand side of the equation.) Since the the membrane is clamped on the boundary,

$$u\big|_{\tilde{Q}} = 0. \tag{10}$$

Since the equation is second-order in time, we need to prescribe both u and u_t at $t = 0$. We thus have the initial-boundary value problem for the wave equation

$$(P_5)\begin{cases} u_{tt} - c^2\Delta u = 0, & \text{for } x \in B \text{ and } t > 0, \\ u(x,0) = f(x), \\ u_t(x,0) = g(x), \\ u\big|_{\tilde{Q}} = 0. \end{cases}$$

10. (P_5) is a typical initial-boundary value problem for a hyperbolic equation.

11. Spatial domains are not always bounded. You might, for example, model vibrations on a long string with a pure initial value problem for the wave equation:

$$(P_6)\begin{cases} u_{tt} - c^2 u_{xx} = 0, & \text{for } x \in \mathbf{R} \text{ and } t > 0, \\ u(x,0) = f(x), \\ u_t(x,0) = g(x). \end{cases}$$

Another example is the Dirichlet problem in the upper half-plane:

$$(P_7)\begin{cases} w_{xx} + w_{yy} = 0, & \text{for } x \in \mathbf{R} \text{ and } y > 0, \\ w(x,0) = f(x). \end{cases} \tag{11}$$

You might wonder why we prescribe u and u_t in the (P_6) but only w in (P_7). We do this because (P_7) is an elliptic problem modeling a time-independent equilibrium. We have to think of $y = 0$ not as an "initial line" but as the boundary of the spatial region $y > 0$. Thus (11) is a boundary condition, not one of a pair of initial conditions.