# Eigenfunction Expansions 3

**1**. We can extend the notion of self-adjointness to regular boundary value problems of any order. Let $L$ be an $n$th order, linear, differential operator

$$L = \sum_{k=0}^{n} a_k(x) D^k, \tag{1}$$

whose the coefficients $a_k$ are smooth and complex-valued, with $a_n(x) \neq 0$ on $[a, b]$. Let $B_1, \ldots, B_n$ be linear boundary operators of order at most $n-1$. Consider the boundary value problem

$$(P_0)\begin{cases} LX = \lambda X & \text{for } a < x < b, \\ B_i X = 0 & \text{for } i = 1, \ldots, n. \end{cases}$$

Integration by parts yields

$$\int_a^b LX(x)\bar{Y}(x)\,dx = \mathcal{B}(X, \bar{Y}) + \int_a^b X(x)\overline{L^*Y(x)}\,dx, \tag{2}$$

where $\mathcal{B}(X, \bar{Y})$ represents the boundary terms and

$$L^*Y = \sum_{k=0}^{n} (-1)^k D^k(\bar{a}_k(x)Y), \tag{3}$$

is the formal adjoint of $L$. In terms of the inner product, this is

$$\langle LX\,,\,Y\rangle = \mathcal{B}(X, \bar{Y}) + \langle X\,,\,L^*Y\rangle\,. \tag{4}$$

**2**. If

$$L = L^*, \quad \text{and} \quad \mathcal{B}(X, \bar{Y}) = 0,$$

for all $X$ and $Y$ in the domain of $L$ (i.e. for all smooth $X$ and $Y$ satisfying the boundary conditions), then

$$\langle LX\,,\,Y\rangle = \langle X\,,\,LY\rangle\,. \tag{5}$$

When this is the case, the problem is called *self-adjoint.*

**3**. The familiar results from the second-order case still pertain.

**a**. The eigenvalues of a self-adjoint problem are real.

**b**. Let $(P_0)$ be self-adjoint, and $\lambda \neq \mu$ be eigenvalues. If $X$ and $Y$ are eigenvectors belonging to $\lambda$ and $\mu$ respectively, then $\langle X\,,\,Y\rangle = 0$. (Thus the eigenspaces of $\lambda$ and $\mu$ are orthogonal.) Each eigenspace can have dimension at most $n$.

**c**. Let $(P_0)$ be self-adjoint, and have eigenvalues $\{\lambda_k\}$ with associated eigenspaces $\{E_{\lambda_k}\}$. Let $\mathcal{O}_k$ be an orthonormal basis for $E_{\lambda_k}$. Then

$$\mathcal{O} = \bigcup_k \mathcal{O}_k,$$

is an orthonormal basis for $L^2[a,b]$.

**4**. **Example**: Let $L$ be the Dirac operator,

$$L = \frac{1}{i}\frac{d}{dx},$$

and $B$ the boundary operator

$$BX = X(0) - X(1).$$

The problem

$$(P_1)\begin{cases} LX = \lambda X & \text{for } 0 < x < 1, \\ BX = 0, \end{cases}$$

is self-adjoint.

**5**. The eigenvalues of $(P_1)$ are

$$\lambda_k = 2\pi i k \quad \text{for} \quad k = 0, \pm 1, \pm 2, \ldots$$

The eigenspaces are all one-dimensional, each with an orthonormal basis consisting of a single eigenfunction

$$e_k(x) = e^{2\pi i k x}.$$

Thus,

$$\mathcal{O} = \{e^{2\pi i k x}\}_{k \in \mathbf{Z}},$$

is an orthonormal basis of $L^2[0,1]$. A function $f \in L^2[0,1]$ has the Fourier expansion

$$f = \sum_{k=-\infty}^{\infty} \hat{f}(k) e_k,$$

where

$$\hat{f}(k) = \int_0^1 f(x) e^{-2\pi i k x}\, dx.$$