Eigenfunction Expansions 2

1. Let $[a, b]$ be a finite interval. Define the second-oder, linear differential operator

$$L = a_2(x)D^2 + a_1(x)D + a_0,$$

where the a_i are smooth and complex-valued, and $a_2(x) \neq 0$ on $[a, b]$. Let B_1 and B_2 be linear boundary operators of at most the first order. If

$$(P_0) \begin{cases} LX = \lambda X, & \text{for } a < x < b, \\ B_1 X = 0, \\ B_2 X = 0, \end{cases}$$

is self-adjoint, it is called a regular Sturm-Liouville problem. We can use such a problem to generate an orthonormal basis for $L^2[a, b]$. The recipe is

a. Find the eigenvalues $\{\lambda_n\}$.

b. For each eigenvalue, determine the eigenspace E_{λ_n}. (Note that $\dim E_{\lambda_n} \leq 2$.)

c. Find an orthonormal basis $\mathcal{O}_n$ of E_{λ_n}.

d. Since the problem is self-adjoint, the sets $\mathcal{O}_n$ are mutually orthogonal. Hence,

$$\mathcal{O} = \bigcup_n \mathcal{O}_n,$$

is itself an orthonormal set. For a regular Sturm-Liouville problem like (P_0), one can show that $\mathcal{O}$ is actually an orthonormal basis of $L^2[a, b]$. Thus, if

$$\mathcal{O} = \{e_n\},$$

and f is in $L^2[a, b]$ then

$$f = \sum_n \langle f \, , \, e_n \rangle \, e_n,$$

where equality is in the sense of L^2.

2. For fixed λ, let $X_1(x, \lambda)$ and $X_2(x, \lambda)$ be linearly independent solutions to the ODE

$$LX - \lambda X = 0, \tag{1}$$

on (a, b). Then every solution $X(x, \lambda)$ to (1) has the form

$$X(x, \lambda) = c_1 X_1(x, \lambda) + c_2 X_2(x, \lambda), \tag{2}$$

for constants c_1 and c_2. Let

$$c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

and $B(\lambda)$ the 2×2 matrix

$$B(\lambda) = \begin{bmatrix} B_1 X_1(\lambda) & B_1 X_2(\lambda) \\ B_2 X_1(\lambda) & B_2 X_2(\lambda) \end{bmatrix}. \tag{3}$$

We apply the boundary operators to X. Since they are linear,

$$B_1 X(\lambda) = c_1 B_1 X_1(\lambda) + c_2 B_1 X_2(\lambda),$$

and

$$B_2 X(\lambda) = c_1 B_2 X_1(\lambda) + c_2 B_2 X_2(\lambda).$$

Thus,

$$\begin{bmatrix} B_1 X(\lambda) \\ B_2 X(\lambda) \end{bmatrix} = B(\lambda) c. \tag{4}$$

3. **Proposition**: The scalar λ is an eigenvalue if and only if

$$\det B(\lambda) = 0. \tag{5}$$

Proof: Suppose that (5) holds for some λ. This implies that there is a *nonzero* vector $c = [\, c_1 \quad c_2 \,]'$ such that

$$B(\lambda) c = 0. \tag{6}$$

By (6),

$$X(x, \lambda) = c_1 X_1(x, \lambda) + c_2 X_2(x, \lambda),$$

satisfies the boundary conditions. And since $c \neq 0$, X is nontrivial. Hence λ is an eigenvalue. To prove the converse, let λ be an eigenvalue. Then by definition, there is a nontrivial solution $X(x, \lambda)$ to (P_0). This solution must have the form (2) for coefficients c_1 and c_2. Since $X(x, \lambda)$ is nontrivial, $c \neq 0$. But as $X(\xi, \lambda)$ satisifes the boundary conditions, c must satisfy (6). Hence the matrix $B(\lambda)$ has a nontrivial kernel, which in turn implies condition (5).

4. The above proposition gives us a simple algebric procedure for finding the eigenvalues of (P_0). Note that the problem need not be self-adjoint. If the problem is self-adjoint, we can confine our search for eigenvalues to the real line. It is sometimes possible to simplify the search still further. Suppose, for example, that the problem is self-adjoint, and that $L = D^2$. Let λ be an eigenvalue with the nontrivial eigenfunction X. Integration by parts shows that

$$\lambda = -\frac{\|X'\|_2}{\|X\|_2}.$$

Thus the eigenvalues are nonpositive.

5. **Example**: Consider the self-adjoint problem

$$(P_1)\begin{cases} X'' = \lambda X, & \text{for } 0 < x < 1, \\ X(0) - X(1) = 0, & \\ X'(0) - X'(1) = 0. & \end{cases}$$

By the preceeding paragraph, we know that the eigenvalues are real and nonpositive. We thus set

$$\lambda = -k^2,$$

for $k > 0$. The equation becomes

$$X'' + k^2 X = 0,$$

with general solution

$$X = \begin{cases} c_1 + c_2 x & \text{for } k = 0, \\ c_1 \cos kx + c_2 \sin kx & \text{for } k > 0. \end{cases}$$

It is convenient to think of B as a function of k. When $k = 0$, we take as independent solutions $X_1 = 1$ and $X_2 = x$. Then

$$B(0) = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix},$$

with

$$\det B(0) = 0.$$

Thus $\lambda_0 = 0$ is an eigenvalue. Any eigenfunction of λ_0 must have the form $X = c_1 + c_2 x$. In order to satisfy the first boundary condition we must set $c_2 = 0$. Hence the eigenspace E_{λ_0} is one-dimensional and has the function

$$c_0(x) \equiv 1,$$

as an orthonormal basis. For $k \neq 0$, we take as independent solutions $X_1 = \cos kx$ and $X_2 = \sin kx$. In this case,

$$B(k) = \begin{bmatrix} 1 - \cos k & -\sin k \\ k \sin k & k(1 - \cos k) \end{bmatrix},$$

with

$$\det B(k) = 2k(1 - \cos k).$$

Thus the nonzero eigenvalues are

$$\lambda_n = -4n^2\pi^2, \quad n = 1, 2, 3, \ldots$$

The corresponding eigenfunctions must be of the form $X = c_1 \cos(2\pi nx) + c_2 \sin(2\pi nx)$. We conclude that for $n > 0$, the eigenspace E_{λ_n} is two-dimensional with orthonormal basis $\{c_n(x), s_n(x)\}$, where

$$c_n(x) = \sqrt{2}\cos(2\pi nx),$$

and

$$s_n(x) = \sqrt{2}\sin(2\pi nx).$$

Thus,

$$\mathcal{O} = \{c_0, c_1, s_1, c_2, s_2, \ldots, c_n, s_n, \ldots\}$$

is an orthonormal basis for $L^2[0,1]$.

6. For problem (P_1), you can use complex exponentials instead of sines and cosines. The eigenfunction belonging to $\lambda_0 = 0$ is of course

$$e_0(x) \equiv 1.$$

For $n > 0$,

$$e_n(x) = e^{2\pi inx} \quad \text{and} \quad e_{-n}(x) = e^{-2\pi inx},$$

form an orthonormal basis for the eigensapce E_{λ_n}. (When checking this, don't forget the complex conjugate in the inner product.) Thus,

$$\mathcal{O} = \{\ldots, e_2, e_1, e_0, e_1, e_2, \ldots\}, \tag{7}$$

is an orthonormal basis of $[0,1]$. If f lies in $L^2[0,1]$,

$$f = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi inx}, \tag{8}$$

where

$$\hat{f}(n) = \langle f\,,\, e_n\rangle = \int_0^1 f(x) e^{-2\pi inx}\, dx, \tag{9}$$

is the nth Fourier coefficent with respect to the basis (7). As usual, it should be understood that (8) holds in the sense of $L^2[0,1]$, *viz*

$$\| f - \sum_{n=-k}^{m} \hat{f}(n) e_n \|_2 \to 0 \quad \text{as} \quad k, m \to \infty.$$