Eigenfunction Expansions 1

1. **Example**: Let $u(x,t)$ be the density of a gas in a straight, narrow, cylindrical tube of length 1. Let f be the initial density. Assume that the ends of the tube are plugged. Thus u satisfies the initial-boundary value problem with "no-flux" boundary conditions:

$$(P_0)\begin{cases} u_t - ku_{xx} = 0, & \text{for } 0 < x < 1,\, t > 0, \\ u(x,0) = f(x), \\ u_x(0,t) = 0, \\ u_x(1,t) = 0. \end{cases}$$

We separate variables:

$$u(x,t) = T(t)X(x), \tag{1}$$

and see that

$$T'(t) = \lambda k T(t), \tag{2}$$

and

$$(P_1)\begin{cases} X'' = \lambda X, & \text{for } 0 < x < 1, \\ X'(0) = 0, \\ X'(1) = 0, \end{cases}$$

for some constant λ. Problem (P_1) is called an eigenvalue problem. Let

$$L = \frac{d^2}{dx^2} = D^2,$$

and define the linear boundary operators

$$B_1X = X'(0),$$

and

$$B_2X = X'(1).$$

With this notation, (P_1) is

$$(P_1)\begin{cases} LX = \lambda X, & \text{for } 0 < x < 1, \\ B_1X = 0, \\ B_2X = 0. \end{cases}$$

Since the first boundary condition involves X only at $x = 0$, and the second, only at $x = 1$, the boundary conditions are called *separated.*

2. **Example**: Suppose now $u(x,t)$ represents the temperature of a thin, insulated, wire ring of circumference 1. Here, the spatial variable x represents arclength along the

ring, measured widdershins (counterclockwise). We thus have the *periodic* boundary conditions

$$u(0,t) = u(1,t), \tag{3}$$

and

$$u_x(0,t) = u_x(1,t). \tag{4}$$

With the initial temperature distribution f, we have the initial boundary value problem

$$(P_2)\begin{cases} u_t - ku_{xx} = 0, & \text{for } 0 < x < 1,\ t > 0, \\ u(x,0) = f(x), \\ u(0,t) - u(1,t) = 0, \\ u_x(0,t) - u_x(1,t) = 0. \end{cases}$$

We set

$$u(x,t) = T(t)X(x),$$

and obtain

$$T'(t) = \lambda k T(t), \tag{5}$$

and

$$(P_3)\begin{cases} X'' = \lambda X, & \text{for } 0 < x < 1, \\ X(0) - X(1) = 0, \\ X'(0) - X'(1) = 0, \end{cases}$$

for some constant λ. If we set

$$L = D^2,$$

and define the linear boundary operators

$$B_1 X = X(0) - v(1),$$

and

$$B_2 X = X'(0) - v'(1),$$

(P_3) becomes

$$(P_3)\begin{cases} LX = \lambda X, \quad \text{for } 0 < x < 1, \\ B_1 X = 0, \\ B_2 X = 0. \end{cases}$$

3. We'll say that a boundary operator B_i is of order k if it contains derivatives up to but not exceeding the kth. In the first example, the both boundary operators are of order 1. In the second, B_1 is of order 0 and B_1 of order 1.

4. Let $[a,b]$ be a finite interval. Define the second-oder, linear differential operator

$$L = a_2(x)D^2 + a_1(x)D + a_0,$$

where the a_i are smooth and complex-valued, and $a_2(x) \neq 0$ on $[a,b]$. Let B_1 and B_2 be linear boundary operators of at most the first order. Consider the problem

$$(P_4) \begin{cases} LX = \lambda X, & \text{for } a < x < b, \\ B_1 X = 0, \\ B_2 X = 0. \end{cases}$$

5. **Note**: The function $X \equiv 0$ is a solution (called the trivial solution) to (P_4). A solution X that is not identically zero is called nontrivial.

6. **Note**: We haven't been specific about the domain of L, that is, the class of functions X on which L operates. We require that

a. X, X' and X'' be in $L^2[a,b]$,

b. $B_1 X = B_2 X = 0$.

For practical purposes, you don't have to worry about (a). Just remember that functions in the domain of L have to satisfy the boundary conditions.

7. **Linear algebraic digression:** Let $A = (a_{ij})$ be a complex, $n \times n$ matrix, and λ a scalar. If the equation

$$AX = \lambda X, \tag{6}$$

has a solution $X \neq 0$, then λ is an *eigenvalue* of A. *Any* vector X satisfying (6) is an *eigenvector* belonging to λ. Note that $X = 0$ (i.e. the zero vector in $\mathbf{C}^n$) is an eigenvector belonging to every eigenvalue.

8. **Proposition**: The set of eigenvectors belonging to an eigenvalue λ is a linear subspace of $\mathbf{C}^n$. (It is called the *eigenspace* of λ.)

9. Let $\langle\, ,\, \rangle$ be the standard inner product on $\mathbf{C}^n$. Let $A^* = (\,\overline{a_{ji}}\,)$ and be the adjoint of A. Then

$$\langle AX\, ,\, Y \rangle = \langle X\, ,\, A^* Y \rangle\, , \tag{7}$$

for all X and Y in $\mathbf{C}^n$.

10. A is called *self-adjoint* if $A = A^*$. If A is self-adjoint then (7) becomes

$$\langle AX\, ,\, Y \rangle = \langle X\, ,\, AY \rangle\, , \tag{8}$$

for all X and Y in $\mathbf{C}^n$. This can be used to prove two important propositions:

11. **Proposition**: The eigenvalues of a self-adjoint matrix are real.

12. **Proposition**: Let A ne a self-adjoint matrix. If X and Y be eigenvectors belonging respectively to the *distinct* eigenvalues μ and λ, then $\langle X\, ,\, Y \rangle = 0$. Thus eigenspaces of distinct eigenvalues of the self-adjoint matrix A are orthogonal.

13. **Definition**: Consider (P_4) with λ *fixed*.

$$(P_4)\begin{cases} LX = \lambda X, & \text{for } a < x < b, \\ B_1 X = 0, \\ B_2 X = 0. \end{cases}$$

If there is a nontrival solution to this problem, then λ is called an eigenvalue. *Any* solution is called an eigenfunction belonging to λ. Note that the trivial solution $X \equiv 0$ is an eigenfunction of every eigenvalue.

14. **Proposition**: The set of eigenfunctions belonging to an eigenvalue λ forms a vector space. (This is called the eigenspace of λ. It is a subspace of $L^2[a, b]$.)

15. **Self-Adjoint Problems:** Integration by parts yields

$$\int_a^b LX(x)\bar{Y}(x)\,dx = \mathcal{B}(X, \bar{Y}) + \int_a^b X(x)\overline{L^*Y}(x)\,dx, \tag{9}$$

where $\mathcal{B}(X, \bar{Y})$ represents the boundary terms and

$$L^*Y = (\bar{a}_2 Y)'' - (\bar{a}_1 Y)' + \bar{a}_0 Y, \tag{10}$$

is the formal adjoint of L. If $\langle \, , \, \rangle$ is the L^2 inner product on $[a, b]$, then (9) becomes

$$\langle LX \, , \, Y \rangle = \mathcal{B}(X, \bar{Y}) + \langle X \, , \, L^*Y \rangle \, . \tag{11}$$

16. If

$$L = L^*, \quad \text{and} \quad \mathcal{B}(X, \bar{Y}) = 0,$$

for all X and Y in the domain of L, then

$$\langle LX \, , \, Y \rangle = \langle X \, , \, LY \rangle \, . \tag{12}$$

When this is the case, the problem is called *self-adjoint.*

17. **Example**: Problems (P_1) and (P_3) are self-adjoint.

18. **Proposition**: The eigenvalues of a self-adjoint problem are real.

19. **Proposition**: Let (P_4) be self-adjoint. and let $\lambda \neq \mu$ be eigenvalues. If Y and Z are eigenvectors belonging to λ and μ respectively, then $\langle Y \, , \, Z \rangle = 0$. (Thus the eigenspaces of λ and μ are orthogonal.)

20. If the coefficient functions a_i are *real-valued* then $L^* = L$ if and only if

$$LX = (a_2 X')' + a_0 X.$$

We usually set $a_2 = -p$ and $a_0 = q$ and write the operator in Sturm-Liouville form:

$$LX = -(pX')' + qX.$$